

The University of Chicago

**CONJUGATE NETS OF
RULED SURFACES IN
A CONGRUENCE**

A DISSERTATION SUBMITTED TO THE
GRADUATE FACULTY IN CANDIDACY FOR
THE DEGREE OF DOCTOR OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS

By
GORDON RICHARD MAGEE

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1. INTRODUCTION

Correspondences which exist between points of a line and planes through this line have played an important rôle in the geometry of ruled space. Koenigs has made a study of one-to-one correspondences of this type in which the anharmonic ratio of any point and a triad of fixed points on a line is equal to the anharmonic ratio of the four corresponding planes; such a correlation is said to be anharmonic.* In general, two anharmonic correlations on the same line have in common two "couples", a couple being composed of a point and a plane in united position. For example, consider a net of ruled surfaces. This consists of two one-parameter families of ruled surfaces in a congruence, such that any pair of ruled surfaces, R_1 and R_2 , one from each family, has a generator in common, at all points of which, with the exception of the focal points, the planes tangent to R_1 and R_2 are distinct. The two correlations of Chasles, relative to the common generator of R_1 and R_2 , constitute a pair of anharmonic correlations, the common pair of couples consisting of the two focal points of the generator and the planes tangent to the focal surfaces at those points. If the planes tangent to R_1 and R_2 at each point of the common generator are divided harmonically by the focal planes of that generator, or dually, if the points of tangency with R_1 and R_2 , of each plane through the common

*G. Koenigs, Sur les Propriétés Infinitésimales de l'Espace Réglé, Annales Scientifiques de l'Ecole Normale Supérieure, Deuxième Série, t. XI (1882), pp. 226-338.

generator, are separated harmonically by the focal points of the generator, then the two correlations of Chasles are said to be in involution or orthogonal.* A net of ruled surfaces which possesses this property shall be called a conjugate net.

It will be assumed throughout this paper that the congruence of generators has distinct focal surfaces; hence the focal planes of any generator will be distinct. The planes tangent to R_1 and R_2 at any point P of the common generator will coincide if, and only if, P is a focal point of that generator, in which case the common tangent plane of R_1 and R_2 is the corresponding focal plane of the generator.

Wilczynski has defined a net of ruled surfaces, R_1 and R_2 , to be a conjugate net if the osculating quadrics of R_1 and R_2 are so related that the planes tangent to these quadrics at each point of the common generator are divided harmonically by the focal planes of the generator.† Evidently the condition of orthogonality of the two correlations of Chasles is equivalent to Wilczynski's definition.

In the Wilczynski definition of a conjugate net, a one-parameter family of non-developable ruled surfaces is defined by means of a completely integrable system of four homogeneous linear partial differential equations; from this system a second system is obtained which defines a second one-parameter

*Koenigs, loc. cit., p. 229.

†E. J. Wilczynski, One-Parameter Families and Nets of Ruled Surfaces, and a New Theory of Congruences. Transactions of the American Mathematical Society, Vol. 21 (1920). This paper will hereafter be referred to as Wilczynski, One-Parameter Families.

family of non-developable ruled surfaces, the two families constituting a net. The point of view of the present paper is to think of a net of ruled surfaces as being generated by the motion of a line of a congruence as the line traces out the curves of a net on a transversal surface of the congruence.

A congruence is defined in §2 by means of a system of five homogeneous linear partial differential equations with two dependent variables and two independent variables; with a different notation, the system was suggested by Green as a basis for a study of the projective differential geometry of surfaces.* It is convenient to refer to this system as system (G).

Section 3 exhibits the effect on system (G) of a transformation of dependent variables and a transformation of independent variables. These transformations lead to two canonical forms for system (G), which are presented in §4. The purpose of §5 is to show that four equations of system (G) give rise to the system which Wilczynski employed in defining a one-parameter family of non-developable ruled surfaces.

A necessary and sufficient condition that any net of ruled surfaces be conjugate is stated in §6. In §7 it is observed that the conjugacy of a net of ruled surfaces is equivalent to the apolarity of the quadric surfaces which osculate a pair of ruled surfaces of the net.

*G. M. Green, Memoir on the General Theory of Surfaces and Rectilinear Congruences, Transactions of the American Mathematical Society, Vol. 20 (1919), p. 149. Hereafter referred to as Green, Surfaces.

Conjugate nets of ruled surfaces in a pair of Green-reciprocal congruences are discussed in §8. A covariant conjugate net of ruled surfaces is generated by the projective normal through a point of a non-ruled surface, as the point traces out the asymptotic curves on this surface; the Green-reciprocal of this net is also conjugate.

A pair of reciprocal congruences represents a special case of a pair of congruences whose generators are skew lines in one-to-one correspondence. Correspondences of this type have been discussed by Fubini and A. J. Cook, who in this connection have studied the question of the existence of a one-parameter family of surfaces which have been called by Cook the intersector surfaces of one congruence with respect to the other. In §9 the theory of the intersector surfaces of a congruence with respect to its Green-reciprocal congruence is linked up with the theory of conjugate nets of ruled surfaces in these congruences. A second particular type of a pair of congruences with generators in one-to-one correspondence is considered in §10; corresponding generators in this case are reciprocal polars with respect to one of the associated quadrics of Lane.

A geometric construction for a covariant transversal surface of a congruence is given in §11, followed in §12 by a geometric determination of a conjugate net of ruled surfaces.

2. THE DEFINING SYSTEM (G) OF DIFFERENTIAL EQUATIONS

In this section we present a system of five homogeneous linear partial differential equations with two dependent and two independent variables, two of the equations of the system being of the first order, and three of the second order. Subject to the conditions of complete integrability, this system defines a rectilinear congruence with respect to the group of projective transformations. The defining system is symmetrical in the independent variables. A second system, which is symmetric with the first system in the dependent variables, is readily obtained from the latter.

Let $x^{(k)}$ and $y^{(k)}$, ($k = 1, 2, 3, 4$), be the homogeneous coordinates of two distinct points, P_x and P_y , in a space of three dimensions. Two surfaces, S_x and S_y , are defined by the equations

$$x^{(k)} = x^{(k)}(u, v) \quad , \quad y^{(k)} = y^{(k)}(u, v) \quad ,$$

where $x^{(k)}$, $y^{(k)}$ are analytic functions of two independent variables u and v . It shall be assumed that both S_x and S_y are proper surfaces, that is, neither surface degenerates into a curve or a fixed point. The points of these two surfaces shall be in one-to-one correspondence, corresponding points possessing the same curvilinear coordinates. The totality of lines ℓ_{xy} which join corresponding points P_x and P_y will form a congruence Γ_{xy} . Further, let us assume that neither S_x nor S_y is a focal surface of Γ_{xy} ; then the determinants

$$(x, x_u, x_v, y)$$

and

$$(y, y_u, y_v, x)$$

will not vanish over the region in which all of the functions $x^{(k)}, y^{(k)}$ exist. Consequently there exists a system of differential equations of the following form, to be known as system (G), which defines, in a sense to be explained presently, the general congruence Γ_{xy} :

$$(G) \quad \begin{aligned} x_{uu} &= p x + \alpha x_u + \beta x_v + L y, \\ x_{uv} &= c x + a x_u + b x_v + M y, \\ x_{vv} &= q x + \gamma x_u + \delta x_v + N y, \\ y_u &= f x + m x_u + s x_v + A y, \\ y_v &= g x + t x_u + n x_v + B y, \end{aligned}$$

in which

$$\Delta = mn - st \neq 0.$$

System (G) represents an analogue of the Gauss equations* which are fundamental in the metric differential geometry of surfaces.

Certain relations exist among the coefficients of system (G) under the hypothesis that the system has a fundamental set of four linearly independent solutions $x^{(k)}$ and four linearly independent solutions $y^{(k)}$ for which the determinant

$$D = (x, x_u, x_v, y)$$

is different from zero. A necessary condition that system (G) have a fundamental set of solutions† is

*L. P. Eisenhart, A treatise on the Differential Geometry of Curves and Surfaces, Ginn & Co. (1909), p. 154.

†G. M. Green, The Linear Dependence of Functions of Several Variables, and Completely Integrable Systems of Homogeneous Linear Partial Differential Equations, Transactions of the American Mathematical Society, Vol. 17 (1916), p. 499.

$$(\chi_{uu})_v = (\chi_{uv})_u,$$

$$(\chi_{uv})_v = (\chi_{vv})_u,$$

$$(\gamma_u)_v = (\gamma_v)_u.$$

These conditions on the coefficients of system (G) yield three equations of the form

$$2\mathcal{A}\chi + \mathcal{B}\chi_u + \mathcal{C}\chi_v + \mathcal{D}\gamma = 0.$$

Since the determinant D is different from zero, the coefficients $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$, in each of the three equations, must be identically zero. We thus obtain, after Green, the following twelve integrability conditions of system (G), which are analogous to the Gauss - Codazzi equations* of metric geometry:

$$\begin{aligned} a_u - a_v + a\beta - \beta\gamma + c &= Lt - Mm, \\ b_u - \beta_v + a\beta + b(b-a) - \beta\delta - \rho &= Ln - Ms, \\ c_u - \rho_v + a\rho + c(b-a) - \beta q &= Lg - Mf, \\ M_u - L_v + aL + M(b-a) - \beta N &= LB - MA, \\ \gamma_u - a_v + \gamma a + a(\delta-a) - b\gamma + q &= Mt - Nm, \\ \delta_u - b_v + \gamma\beta - a\delta - c &= Mn - Ns, \\ q_u - c_v + \gamma\rho + c(\delta-a) - bq &= Mg - Nf, \\ N_u - M_v + \gamma L + M(\delta-a) - bN &= MB - NA, \\ t_u - m_v + ta + a(m-m) - s\gamma + g &= At - Bm, \\ m_u - s_v + t\beta + b(m-m) - s\delta - f &= Am - Bs, \\ g_u - f_v + t\rho + c(m-m) - sq &= Ag - Bf, \\ B_u - A_v + tL + M(m-m) - sN &= 0. \end{aligned}$$

*Eisenhart, loc. cit., pp. 155-156.

If we add the first, sixth, and last integrability conditions, we obtain the relation

$$(a + \delta + B)_u = (b + \alpha + A)_v.$$

Hence there exists a function $F(u, v)$ such that

$$F_u = b + \alpha + A,$$

$$F_v = a + \delta + B,$$

and on differentiating the determinant D we have

$$D_u = D F_u, \quad D_v = D F_v,$$

so that

$$D = C e^F,$$

where C denotes a non-vanishing constant.

Since the determinant Δ is different from zero, we may solve the fourth and fifth equations of system (G) for x_u and x_v as linear combinations of y, y_u, y_v, x . Then by differentiating the fourth and fifth equations of system (G) with respect to u and v and making use of the first three equations of that system along with the equations which express x_u and x_v as linear combinations of y, y_u, y_v, x , we obtain y_{uu}, y_{uv} , and y_{vv} each in terms of y, y_u, y_v, x . Thus we have the following system $(\bar{G})$, which is symmetric with system (G) in x and y :

$$\begin{aligned}
 y_{uu} &= \bar{p} y + \bar{a} y_u + \bar{b} y_v + \bar{L} x, \\
 y_{uv} &= \bar{c} y + \bar{a} y_u + \bar{b} y_v + \bar{M} x, \\
 y_{vv} &= \bar{q} y + \bar{\gamma} y_u + \bar{\delta} y_v + \bar{N} x, \\
 x_u &= \bar{f} y + \bar{m} y_u + \bar{s} y_v + \bar{A} x, \\
 x_v &= \bar{g} y + \bar{t} y_u + \bar{n} y_v + \bar{B} x.
 \end{aligned}
 \tag{\bar{G}}$$

We find

$$(1) \quad \begin{aligned} \bar{f} &= (\delta B - m A) / \Delta, \quad \bar{m} = n / \Delta, \quad \bar{s} = -s / \Delta, \quad \bar{A} = (\delta g - m f) / \Delta, \\ \bar{g} &= (t A - m B) / \Delta, \quad \bar{t} = -t / \Delta, \quad \bar{n} = m / \Delta, \quad \bar{B} = (t f - m g) / \Delta. \end{aligned}$$

It is convenient to define the following functions:

$$\begin{aligned} g_{11} &= m a + s a + m_u + f, & g_{44} &= n \delta + t b + n_v + g, \\ g_{12} &= m \beta + s b + s_u, & g_{43} &= n \gamma + t a + t_v, \\ g_{13} &= m \mu + s c + f_u, & g_{42} &= n g + t c + g_v, \\ g_{14} &= m M + s N, & g_{41} &= n M + t L, \\ g_{21} &= m a + s \gamma + m_v, & g_{34} &= n b + t \beta + n_u, \\ g_{22} &= m b + s \delta + s_v + f, & g_{33} &= n a + t d + t_u + g, \\ g_{23} &= m c + s g + f_v, & g_{32} &= n c + t \mu + g_u, \\ g_{24} &= m L + s M, & g_{31} &= n N + t M. \end{aligned}$$

Then

$$(2) \quad \begin{aligned} \bar{f} &= A_u + \bar{f} g_{11} + \bar{g} g_{12} + g_{24}, \\ \bar{a} &= A + \bar{m} g_{11} + \bar{t} g_{12}, \\ \bar{\beta} &= \bar{s} g_{11} + \bar{m} g_{12}, \\ \bar{L} &= \bar{A} g_{11} + \bar{B} g_{12} + g_{13}, \\ \bar{c} &= A_v + \bar{f} g_{21} + \bar{g} g_{22} + g_{14} = B_u + \bar{f} g_{33} + \bar{g} g_{34} + g_{41}, \\ \bar{a} &= \bar{m} g_{21} + \bar{t} g_{22} = B + \bar{m} g_{33} + \bar{t} g_{34}, \\ \bar{b} &= A + \bar{s} g_{21} + \bar{m} g_{22} = \bar{s} g_{33} + \bar{m} g_{34}, \\ \bar{M} &= \bar{A} g_{21} + \bar{B} g_{22} + g_{23} = \bar{A} g_{33} + \bar{B} g_{34} + g_{32}, \\ \bar{g} &= B_v + \bar{f} g_{43} + \bar{g} g_{44} + g_{31}, \\ \bar{\gamma} &= \bar{m} g_{43} + \bar{t} g_{44}, \\ \bar{S} &= B + \bar{s} g_{43} + \bar{m} g_{44}, \\ \bar{N} &= \bar{A} g_{43} + \bar{B} g_{44} + g_{42}. \end{aligned}$$

The following substitution group provides a check on the calculation of the coefficients of system ($\bar{G}$):

$$\begin{pmatrix} u & p & a & \beta & L & c & a & M & f & m & s & A & 1 & \bar{a} \\ v & q & \delta & \gamma & N & c & b & M & g & n & t & B & 4 & 3 \end{pmatrix}.$$

In passing, we note that

$$\bar{m}\bar{n} - \bar{s}\bar{t} = 1/\Delta,$$

$$\bar{a} + \bar{s} + \bar{B} = a + \delta + B + (\log \Delta)_v,$$

$$\bar{b} + \bar{a} + \bar{A} = b + a + A + (\log \Delta)_u,$$

and also

$$(y, y_u, y_v, x) = -\Delta(x, x_u, x_v, y).$$

System (G) was set up on the hypothesis that a fundamental set of solutions $x^{(k)}$, $y^{(k)}$, ($k = 1, 2, 3, 4$), was known, which were analytic throughout a region of definition of the variables u and v , and for which two determinants were different from zero throughout this region. A necessary condition that system (G) have a fundamental set of solutions of this type is that the coefficients of the system satisfy the integrability conditions identically. The system is then completely integrable, and the most general pair of analytic functions (x, y) which satisfy system (G) is

$$x = \sum_{k=1}^4 c_k x^{(k)}, \quad y = \sum_{k=1}^4 c_k y^{(k)},$$

where $c_1, \dots, c_4$ are arbitrary constants. Consequently system (G) determines not only two surfaces, S_x and S_y , with points in one-to-one correspondence but all projective transformations of these surfaces. It is in this sense that the congruence Γ_{xy} is characterized by the completely integrable system (G).

3. TRANSFORMATIONS OF SYSTEM (G)

System (G) is not uniquely determined when the congruence Γ_{xy} is given. It is the purpose of this section to study the effect on system (G) of the replacement of the reference surfaces S_x, S_y by an arbitrary pair of transversal surfaces of the congruence. Next, the effect of a change of proportionality factor in the homogeneous coordinate system is observed; the latter transformation is, of course, a special case of the former. Finally a transformation of the independent variables u and v is made. In each of these cases, the form of system (G) is invariant.

The transformation

$$(3) \quad \begin{aligned} \chi &= \lambda \xi + \tau \eta, \\ \eta &= \sigma \xi + \mu \eta, \end{aligned}$$

where $\lambda, \tau, \sigma, \mu$ are arbitrary functions of u and v , for which

$$\lambda\mu - \sigma\tau \neq 0, \quad \tau\sigma \neq 0,$$

is the most general transformation which leaves each line of the congruence Γ_{xy} unaltered but replaces the surfaces of reference, S_x and S_y , by two arbitrary surfaces, S_ξ and S_η .

Under a transformation of this form, system (G) becomes:

$$(4) \quad \begin{aligned} \lambda \xi_{uu} + \tau \eta_{uu} &= (\rho\lambda + \alpha\lambda_u + \beta\lambda_v + L\sigma - \lambda_{uu})\xi + (\alpha\lambda - 2\lambda_u)\xi_u + \beta\lambda\xi_v \\ &\quad + (\rho\tau + \alpha\tau_u + \beta\tau_v + L\mu - \tau_{uu})\eta + (\alpha\tau - 2\tau_u)\eta_u + \beta\tau\eta_v, \\ \lambda \xi_{uv} + \tau \eta_{uv} &= (\rho\lambda + \alpha\lambda_u + \beta\lambda_v + M\sigma - \lambda_{uv})\xi + (\alpha\lambda - \lambda_v)\xi_u + (\beta\lambda - \lambda_u)\xi_v \\ &\quad + (\rho\tau + \alpha\tau_u + \beta\tau_v + M\mu - \tau_{uv})\eta + (\alpha\tau - \tau_v)\eta_u + (\beta\tau - \tau_u)\eta_v, \\ \lambda \xi_{vv} + \tau \eta_{vv} &= (\rho\lambda + \gamma\lambda_u + \delta\lambda_v + N\sigma - \lambda_{vv})\xi + \gamma\lambda\xi_u + (\delta\lambda - 2\lambda_v)\xi_v \\ &\quad + (\rho\tau + \gamma\tau_u + \delta\tau_v + N\mu - \tau_{vv})\eta + \gamma\tau\eta_u + (\delta\tau - 2\tau_v)\eta_v, \end{aligned}$$

$$\begin{aligned}
 (\mu - m\tau)\eta_u - s\tau\eta_v &= (f\lambda + m\lambda_u + s\lambda_v + A\sigma - \sigma_u)\xi + (m\lambda - \sigma)\xi_u + s\lambda\xi_v \\
 &\quad + (f\tau + m\tau_u + s\tau_v + A\mu - \mu_u)\eta, \\
 -t\tau\eta_u + (\mu - m\tau)\eta_v &= (g\lambda + t\lambda_u + m\lambda_v + B\sigma - \sigma_v)\xi + t\lambda\xi_u + (m\lambda - \sigma)\xi_v \\
 &\quad + (g\tau + t\tau_u + m\tau_v + B\mu - \mu_v)\eta.
 \end{aligned}$$

The focal points of a generator ℓ_{xy} of the congruence Γ_{xy} are given by $x + \omega y$, where ω satisfies the quadratic equation

$$E(\omega) = \Delta\omega^2 + (m+n)\omega + 1 = 0,$$

whose roots are assumed to be distinct, so that

$$\Omega^2 \equiv (m-n)^2 + 4s\tau \neq 0.$$

The determinant of the coefficients of η_u and η_v in the fourth and fifth equations of system (4) is given by

$$E' \equiv \mu^2 E(-\tau/\mu).$$

Hence if ξ is not a focal point of ℓ_{xy} we may solve for η_u and η_v in terms of ξ, ξ_u, ξ_v, η :

$$\begin{aligned}
 \eta_u &= f'\xi + m'\xi_u + s'\xi_v + A'\eta, \\
 \eta_v &= g'\xi + t'\xi_u + m'\xi_v + B'\eta,
 \end{aligned}
 \tag{5}$$

in which the following coefficients are of special interest:

$$\begin{aligned}
 E'm' &= m\lambda\mu + m\sigma\tau - \sigma\mu - \Delta\lambda\tau, \\
 E'n' &= m\sigma\tau + m\lambda\mu - \sigma\mu - \Delta\lambda\tau, \\
 E's' &= (\lambda\mu - \sigma\tau)s, \\
 E't' &= (\lambda\mu - \sigma\tau)t.
 \end{aligned}
 \tag{6}$$

It follows that

$$\Delta' \equiv m'n' - s't' = \sigma^2 E(-\lambda/\sigma)/E'.$$

Therefore $\Delta' \neq 0$ if η is not a focal point of ℓ_{xy} .

The integrability conditions of the transformed system would impose the restriction

$$(7) \quad (\eta_u)_v = (\eta_v)_u.$$

When the expressions (5) for η_u and η_v are substituted in the first three equations of (4), we obtain the equations

$$(8) \quad \begin{aligned} (\lambda + \tau m') \xi_{uu} + \tau \sigma' \xi_{uv} &= A_{11} \xi + A_{12} \xi_u + A_{13} \xi_v + A_{14} \eta, \\ (\lambda + \tau m') \xi_{uv} + \tau \sigma' \xi_{vv} &= A_{21} \xi + A_{22} \xi_u + A_{23} \xi_v + A_{24} \eta, \\ \tau t' \xi_{uv} + (\lambda + \tau m') \xi_{vv} &= A_{31} \xi + A_{32} \xi_u + A_{33} \xi_v + A_{34} \eta, \end{aligned}$$

which are equivalent, in view of (7), to the three equations

$$(8') \quad \begin{aligned} (\lambda + \tau m') \xi_{uu} + \tau \sigma' \xi_{uv} &= A_{11} \xi + A_{12} \xi_u + A_{13} \xi_v + A_{14} \eta, \\ \tau t' \xi_{uu} + (\lambda + \tau m') \xi_{uv} &= B_{21} \xi + B_{22} \xi_u + B_{23} \xi_v + B_{24} \eta, \\ \tau t' \xi_{uv} + (\lambda + \tau m') \xi_{vv} &= A_{31} \xi + A_{32} \xi_u + A_{33} \xi_v + A_{34} \eta, \end{aligned}$$

where A_{ij} , B_{2j} , ($i = 1, 2, 3$; $j = 1, 2, 3, 4$), are functions of the coefficients of system (G) and of λ , μ , σ , τ . The determinants of the coefficients of ξ_{uu} , ξ_{uv} , ξ_{vv} in (8) and (8') are respectively

$$(\lambda + \tau m') \{ \Delta' \tau^2 + (m' + n') \tau \lambda + \lambda^2 \}$$

and

$$(\lambda + \tau m') \{ \Delta' \tau^2 + (m' + m') \tau \lambda + \lambda^2 \}.$$

Since

$$\Delta' \tau^2 + (m' + n') \tau \lambda + \lambda^2 = (\lambda \mu - \sigma \tau)^2 / E',$$

we need only consider the vanishing of the quantities

$$(9) \quad \lambda + \tau m', \quad \lambda + \tau n'.$$

A necessary and sufficient condition that the quantities (9) be simultaneously zero is

$$m = n,$$

in which case

$$\tau:\mu=1:m=1:m,$$

and therefore

$$\xi = m\alpha - \eta = m\alpha - \eta.$$

If, then, in system (G) we put $m = m$ and apply the transformation

$$\begin{aligned}\alpha &= \lambda \xi + \eta, \\ \eta &= \sigma \xi + m \eta,\end{aligned}$$

where

$$m\lambda - \sigma \neq 0, \quad \sigma \neq 0,$$

we find

$$\begin{aligned}f' &= -(q\lambda + t\lambda_u + m\lambda_v + B\sigma - \sigma_v)/t, \\ m' &= -\lambda, \\ s' &= -(m\lambda - \sigma)/t, \\ A' &= -(q + Bm - m_v)/t, \\ g' &= -(f\lambda + m\lambda_u + s\lambda_v + A\sigma - \sigma_u)/s, \\ t' &= -(m\lambda - \sigma)/s, \\ m' &= -\lambda, \\ B' &= -(f + Am - m_u)/s.\end{aligned}$$

From the first and third of equations (8) we obtain

$$\begin{aligned}s' \xi_{uv} &= A_{11} \xi + A_{12} \xi_u + A_{13} \xi_v + A_{14} \eta, \\ t' \xi_{uv} &= A_{31} \xi + A_{32} \xi_u + A_{33} \xi_v + A_{34} \eta,\end{aligned}$$

where

$$\begin{aligned}A_{11} &= -f'_u - A'f' + \alpha f' + \beta g' + \lambda \lambda + \alpha \lambda_u + \beta \lambda_v + L\sigma - \lambda_{uu}, \\ A_{12} &= -m'_u - A'm' - f' + \alpha m' + \beta t' + \alpha \lambda - 2\lambda_u, \\ A_{13} &= -s'_u - A's' + \alpha s' + \beta m' + \beta \lambda,\end{aligned}$$

$$A_{14} = -A'_u - A'^2 + \alpha A' + \beta B' + \mu + Lm ,$$

$$A_{31} = -g'_v - B'g' + \delta g' + \gamma f' + g\lambda + \delta \lambda_v + \gamma \lambda_u + N\sigma - \lambda_{uv} ,$$

$$A_{32} = -t'_v - B't' + \delta t' + \gamma m' + \gamma \lambda ,$$

$$A_{33} = -m'_v - B'm' - g' + \delta m' + \gamma s' + \delta \lambda - 2\lambda_v ,$$

$$A_{34} = -B'_v - B'^2 + \delta B' + \gamma A' + g + Nm .$$

It may be readily verified that

$$t'A_{ij} = s'A_{3j} \quad (j=1,2,3,4),$$

and hence

$$(10) \quad \xi_{uv} = c'\xi + \alpha'\xi_u + b'\xi_v + M'\eta ,$$

where

$$c' = A_{11}/s' = A_{31}/t' ,$$

$$\alpha' = A_{12}/s' = A_{32}/t' ,$$

$$b' = A_{13}/s' = A_{33}/t' ,$$

$$M' = A_{14}/s' = A_{34}/t' .$$

Also, from the second of equations (8') and (8), we find

$$(11) \quad \begin{aligned} \xi_{uu} &= \mu'\xi + \alpha'\xi_u + \beta'\xi_v + L'\eta , \\ \xi_{vv} &= g'\xi + \gamma'\xi_u + \delta'\xi_v + N'\eta , \end{aligned}$$

where

$$\mu' = B_{21}/t' = (-g'_u - B'f' + \alpha f' + b g' + c\lambda + \alpha \lambda_u + b \lambda_v + M\sigma - \lambda_{uv})/t' ,$$

$$\alpha' = B_{22}/t' = (-t'_u - B'm' - g' + b t' - \lambda_v)/t' ,$$

$$\beta' = B_{23}/t' = (\alpha s' - B's')/t' = s(\alpha - B')/t ,$$

$$L' = B_{24}/t' = (-B'_u - A'B' + \alpha A' + b B' + c + Mm)/t' ,$$

$$g' = A_{21}/s' = (-f'_v - A'g' + b g' + \alpha f' + c\lambda + \alpha \lambda_u + b \lambda_v + M\sigma - \lambda_{uv})/s' ,$$

$$\gamma' = A_{22}/s' = (b t' - A't')/s' = t(b - A')/s ,$$

$$\delta' = A_{23}/s' = (-s'_v - A'm' - f' + \alpha s' - \lambda_u)/s' ,$$

$$N' = A_{24}/s' = (-A'_v - A'B' + b B' + \alpha A' + c + Mm)/s' .$$

Thus when the quantities (9) are both zero, a transformation of the form (3) transforms system (G) into a system of differential equations composed of (5), (10) and (11). The form of system (G) is therefore preserved under these circumstances. A geometric characterization of the surface S_f , where

$$\xi = m\alpha - \gamma = m\alpha - \eta,$$

is given in §6. The surface S_h , where

$$\eta = \alpha - \lambda\gamma/\sigma,$$

may be any surface which is distinct from S_f and is not a focal surface of the congruence.

Let us consider now the case in which

$$m \neq m', \quad \lambda + \tau m' = 0.$$

As an immediate consequence we have

$$\xi = m\alpha - \eta,$$

and

$$E' = -\sigma t.$$

If the parametric ruled surfaces of the congruence are non-developable, then both s and t are different from zero; under this assumption, which is tantamount to assuming that $\xi = m\alpha - \eta$ is not a focal point of $\mathcal{L}_{xy}$, we can obtain η_u and η_v in terms of ξ, ξ_u, ξ_v, η . From equations (8') we can also obtain $\xi_{uu}, \xi_{uv}, \xi_{vv}$, each in terms of ξ, ξ_u, ξ_v, η . A similar statement holds for the case in which

$$m \neq m', \quad \lambda + \tau m' = 0,$$

and therefore

$$\xi = m\alpha - \eta,$$

with the exception that we solve equations (8) for $\xi_{uu}, \xi_{uv}, \xi_{vv}$

in terms of ξ, ξ_u, ξ_v, η . In both cases the representation in terms of ξ, ξ_u, ξ_v, η is unique.

Hence any two distinct surfaces, S_ξ and S_η , which are not focal surfaces of the congruence, may be chosen as surfaces of reference; when referred to these surfaces the congruence is characterized by a system of five partial differential equations in ξ and η of the same form as those of system (G), and the two systems are equivalent, in the sense that either one can be transformed into the other by a transformation of the form (3).

A function of the coefficients of system (G) and their derivatives, which remains invariant under a transformation of the form (3), is said to be a seminvariant of the system. It follows from (6) that

$$(n' - m')/s' = (n - m)/s, \quad t'/s' = t/s, \quad \Omega'/s' = \Omega/s.$$

Therefore the functions

$$(n - m)/s, \quad t/s, \quad \Omega/s,$$

and the corresponding functions obtained by interchanging s and t , are absolute seminvariants of system (G).

If $\tau \equiv 0$, $\sigma \equiv 0$, the transformation (3) is of the form

$$(12) \quad x = \lambda(u, v) \bar{x}, \quad y = \mu(u, v) \bar{y}.$$

A transformation of the form (12) represents merely a change of proportionality factor in the coordinate system; under this transformation, the congruence Γ_{xy} and the surfaces of reference S_x, S_y are unchanged, while system (G) goes over directly into a system of the same form, namely:

$$\begin{aligned}\bar{x}_{uu} &= (\rho + \alpha \lambda_u / \lambda + \beta \lambda_v / \lambda - \lambda_{uu} / \lambda) \bar{x} + (\alpha - 2 \lambda_u / \lambda) \bar{x}_u + \beta \bar{x}_v + (L \mu / \lambda) \bar{y}, \\ \bar{x}_{uv} &= (c + a \lambda_u / \lambda + b \lambda_v / \lambda - \lambda_{uv} / \lambda) \bar{x} + (a - \lambda_v / \lambda) \bar{x}_u + (b - \lambda_u / \lambda) \bar{x}_v + (M \mu / \lambda) \bar{y}, \\ \bar{x}_{vv} &= (g + \gamma \lambda_u / \lambda + \delta \lambda_v / \lambda - \lambda_{vv} / \lambda) \bar{x} + \gamma \bar{x}_u + (\delta - 2 \lambda_v / \lambda) \bar{x}_v + (N \mu / \lambda) \bar{y}, \\ \bar{y}_u &= (f + m \lambda_u / \lambda + n \lambda_v / \lambda) \lambda \bar{x} / \mu + (m \lambda / \mu) \bar{x}_u + (n \lambda / \mu) \bar{x}_v + (A - \mu_u / \mu) \bar{y}, \\ \bar{y}_v &= (q + t \lambda_u / \lambda + m \lambda_v / \lambda) \lambda \bar{x} / \mu + (t \lambda / \mu) \bar{x}_u + (m \lambda / \mu) \bar{x}_v + (B - \mu_v / \mu) \bar{y}.\end{aligned}$$

A transformation of the form

$$(13) \quad \bar{u} = \phi(u, v), \quad \bar{v} = \psi(u, v),$$

where ϕ and ψ are arbitrary functions such that

$$\bar{J} \equiv \phi_u \psi_v - \phi_v \psi_u \neq 0,$$

interchanges amongst themselves the lines of the congruence Γ_{xy} .

The result of transformation (13) on system (G) is to produce

a new system of the same form, namely:

$$\begin{aligned}\bar{x}_{\bar{u}\bar{u}} &= \bar{\rho} \bar{x} + \bar{\alpha} \bar{x}_{\bar{u}} + \bar{\beta} \bar{x}_{\bar{v}} + \bar{L} \bar{y}, \\ \bar{x}_{\bar{u}\bar{v}} &= \bar{c} \bar{x} + \bar{a} \bar{x}_{\bar{u}} + \bar{b} \bar{x}_{\bar{v}} + \bar{M} \bar{y}, \\ \bar{x}_{\bar{v}\bar{v}} &= \bar{g} \bar{x} + \bar{\gamma} \bar{x}_{\bar{u}} + \bar{\delta} \bar{x}_{\bar{v}} + \bar{N} \bar{y}, \\ \bar{y}_{\bar{u}} &= \bar{f} \bar{x} + \bar{m} \bar{x}_{\bar{u}} + \bar{n} \bar{x}_{\bar{v}} + \bar{A} \bar{y}, \\ \bar{y}_{\bar{v}} &= \bar{q} \bar{x} + \bar{t} \bar{x}_{\bar{u}} + \bar{m} \bar{x}_{\bar{v}} + \bar{B} \bar{y},\end{aligned}$$

in which

$$\begin{aligned}\bar{J}^2 \bar{\rho} &= \rho \psi_v^2 - 2c \psi_u \psi_v + g \psi_u^2, \\ \bar{J}^2 \bar{\alpha} &= (\alpha \phi_u + \beta \phi_v - \phi_{uu}) \psi_v^2 - 2(\alpha \phi_u + b \phi_v - \phi_{uv}) \psi_u \psi_v + (\gamma \phi_u + \delta \phi_v - \phi_{vv}) \psi_u^2, \\ \bar{J}^2 \bar{\beta} &= (\alpha \psi_u + \beta \psi_v - \psi_{uu}) \psi_v^2 - 2(\alpha \psi_u + b \psi_v - \psi_{uv}) \psi_u \psi_v + (\gamma \psi_u + \delta \psi_v - \psi_{vv}) \psi_u^2, \\ \bar{J}^2 \bar{L} &= L \psi_v^2 - 2M \psi_u \psi_v + N \psi_u^2, \\ \bar{J}^2 \bar{c} &= -\rho \phi_v \psi_v + c(\phi_u \psi_v + \phi_v \psi_u) - g \phi_u \psi_u, \\ \bar{J}^2 \bar{a} &= -(\alpha \phi_u + \beta \phi_v - \phi_{uu}) \phi_v \psi_v + (\alpha \phi_u + b \phi_v - \phi_{uv}) \chi(\phi_u \psi_v + \phi_v \psi_u) - (\gamma \phi_u + \delta \phi_v - \phi_{vv}) \phi_u \psi_u, \\ \bar{J}^2 \bar{b} &= -(\alpha \psi_u + \beta \psi_v - \psi_{uu}) \phi_v \psi_v + (\alpha \psi_u + b \psi_v - \psi_{uv}) \chi(\phi_u \psi_v + \phi_v \psi_u) - (\gamma \psi_u + \delta \psi_v - \psi_{vv}) \phi_u \psi_u.\end{aligned}$$

$$J^2 \bar{M} = -L \phi_{\nu} \psi_{\nu} + M(\phi_u \psi_{\nu} + \phi_{\nu} \psi_u) - N \phi_u \psi_u ,$$

$$J^2 \bar{g} = \mu \phi_{\nu}^2 - 2\kappa \phi_u \phi_{\nu} + g \phi_u^2 ,$$

$$J^2 \bar{f} = (\alpha \phi_u + \beta \phi_{\nu} - \phi_{uu}) \phi_{\nu}^2 - 2(\alpha \phi_u + \beta \phi_{\nu} - \phi_{uu}) \phi_u \phi_{\nu} + (\gamma \phi_u + \delta \phi_{\nu} - \phi_{\nu\nu}) \phi_u^2 ,$$

$$J^2 \bar{\delta} = (\alpha \psi_u + \beta \psi_{\nu} - \psi_{uu}) \phi_{\nu}^2 - 2(\alpha \psi_u + \beta \psi_{\nu} - \psi_{uu}) \phi_u \phi_{\nu} + (\gamma \psi_u + \delta \psi_{\nu} - \psi_{\nu\nu}) \phi_u^2 ,$$

$$J^2 \bar{N} = L \phi_{\nu}^2 - 2M \phi_u \phi_{\nu} + N \phi_u^2 ,$$

$$J \bar{f} = f \psi_{\nu} - g \psi_u ,$$

$$J \bar{m} = (m \phi_u + \Delta \phi_{\nu}) \psi_{\nu} - (t \phi_u + n \phi_{\nu}) \psi_u ,$$

$$J \bar{\Delta} = \Delta \psi_{\nu}^2 + (m - n) \psi_u \psi_{\nu} - t \psi_u^2 ,$$

$$J \bar{A} = A \psi_{\nu} - B \psi_u ,$$

$$J \bar{g} = g \phi_u - f \phi_{\nu} ,$$

$$J \bar{t} = -\{\Delta \phi_{\nu}^2 + (m - n) \phi_u \phi_{\nu} - t \phi_u^2\} ,$$

$$J \bar{n} = (t \psi_u + n \psi_{\nu}) \phi_u - (m \psi_u + \Delta \psi_{\nu}) \phi_{\nu} ,$$

$$J \bar{B} = B \phi_u - A \phi_{\nu} .$$

4. TWO CANONICAL FORMS OF SYSTEM (G)

By means of the transformations (12) and (13) it is possible to obtain two canonical forms of system (G). The first of these canonical forms will be designated system (F), and it defines, except for a projective transformation, a congruence Γ_{xy} whose parametric surfaces intersect the reference surface S_x in the asymptotic curves thereon. The first and third equations of system (F) constitute Fubini's canonical form for the system of equations defining the surface S_x . The second canonical form of system (G) will be designated system (L), and it characterizes a congruence Γ_{xy} whose reference surfaces are in the relation of a fundamental transformation with respect to the congruence.*

To obtain system (F), the functions ϕ and ψ of the transformation (13) are chosen as the roots of the quadratic partial differential equation

$$L\chi_v^2 - 2M\chi_u\chi_v + N\chi_u^2 = 0,$$

in which it is assumed that

$$(14) \quad LN - M^2 \neq 0.$$

For this choice of ϕ and ψ , the coefficients $\bar{L}$ and $\bar{N}$ of the transformed system are both zero and $\bar{M}$ is different from zero. The differential equation of the asymptotic curves on S_x is

$$(15) \quad L du^2 + 2M du dv + N dv^2 = 0,$$

*A third canonical form for system (G) has been obtained by V. G. Grove (Transactions of the American Mathematical Society, Vol. 32 (1930), pp. 473-484.)

so that the condition (14) corresponds to the case in which S_x is non-developable. Consider then system (G) in which
(16) $L=N=0$, $M \neq 0$.

The parametric net of ruled surfaces of the congruence Γ_{xy} thus defined intersects the surface S_x in the asymptotic net on S_x . Let us further suppose that S_x is not a ruled surface; then

$$\beta\gamma \neq 0.$$

From the first, sixth, and last integrability conditions of this system we have

$$(a_u - b_v) - (a_v - b_u) = M(m - m) = A_v - B_u.$$

From the fourth and eighth of these integrability conditions we find that

$$A_v - B_u = (a_v - b_u) + (a_u - b_v).$$

Therefore

$$a_v = b_u.$$

Hence there exists a function $\theta(u, v)$ determined, except for an additive constant, by

$$(17) \quad \theta_u = a, \quad \theta_v = b.$$

By a transformation of the form (12), where

$$\lambda^2 = e^{\theta}/\beta\gamma, \quad \mu = \lambda/M,$$

system (G) goes over into a system of the following form, viz., system (F):

$$(F) \quad \begin{aligned} x_{uu} &= p x + \theta_u x_u + \beta x_v, \\ x_{uv} &= r x + a x_u + b x_v + y, \\ x_{vv} &= q x + \gamma x_u + \theta_v x_v, \\ y_u &= f x + m x_u + s x_v + A y, \\ y_v &= g x + t x_u + n x_v + B y, \end{aligned}$$

where

$$\theta = \log \beta \gamma.$$

The first and third equations of system (F) represent the surface S_X with respect to the group of projective transformations; the coordinates x are the normal coordinates of Fubini. If

$$a = b = 0,$$

then P_Y is a point on the projective normal through P_X of the surface S_X , and the congruence Γ_{XY} becomes the congruence of projective normals of S_X .*

The integrability conditions of system (F) are as follows:

$$\begin{aligned} f &= p_r - a p + \beta q - c_u - c(b - \theta_u), \\ q &= q_u - b q + \gamma p - c_v - c(a - \theta_v), \\ m &= \theta_{uv} - a_u - ab + \beta \gamma - c, \\ n &= \theta_{uv} - b_v - ab + \beta \gamma - c, \\ s &= \beta_v - b_u - a\beta - b(b - \theta_u) + \beta\theta_v + p, \\ t &= \gamma_u - a_v - b\gamma - a(a - \theta_v) + \gamma\theta_u + q, \\ A &= \theta_u - b, \quad f_{32} + Bf = f_{23} + Ag, \\ B &= \theta_v - a, \quad f_{22} + Am = f_{34} + Bs, \\ m - n &= A_v - B_u, \quad f_{33} + Bm = f_{21} + At, \end{aligned}$$

where the functions f_{ij} are obtainable from the corresponding functions g_{ij} of § 2 by means of (16), with $M = 1$, and by

*E. Bompiani, Determinazione varie delle normali proiettive di una superficie, Atti della Reale Accademia dei Lincei, Vol. 9 (1929), p. 40.

means of (17). As a consequence of these integrability conditions we have

$$m-m = a_u - b_v.$$

Hence if Γ_{xy} is the congruence of projective normals of S_x , then $m=n$. A geometric interpretation of the foregoing remark will be given in a later section of this paper. For future use we wish to define ψ_1 and ψ_2 by the formulas

$$\psi_1 = (\log \beta \gamma^2)_u, \quad \psi_2 = (\log \beta^2 \gamma)_v.$$

The most general transformation of the group defined by equations (3) and (13), which leaves system (F) invariant, is

$$x = k\bar{x}, \quad y = k\bar{y}, \quad \bar{u} = U(u), \quad \bar{v} = V(v),$$

where k is an arbitrary constant, and $U'V' = 1$.

The second canonical form of system (G) may be obtained as follows. By a transformation of the form (13), where the functions ϕ and ψ are solutions of the partial differential equations of the first order,

$$2\Delta \phi_v + (m-n-\Omega)\phi_u = 0,$$

$$2\Delta \psi_v + (m-n+\Omega)\psi_u = 0,$$

in which

$$\Omega^2 = (m-n)^2 + 4st \neq 0,$$

we remove from system (G) the coefficients s and t , and consequently the coefficients $\bar{s}$ and $\bar{t}$ of the symmetric system $(\bar{G})$ are zero. The parametric ruled surfaces of the congruence Γ_{xy} are then developables. If the surfaces S_x and S_y are in the relation of a fundamental transformation with respect to Γ_{xy} , we have $M = \bar{M} = 0$, where $\bar{M}$ is the coefficient of x in the

second equation of system $(\bar{G})$. It follows from the last integrability conditions of systems (G) and $(\bar{G})$ that under these circumstances

$$A_{\nu} = B_u, \quad \bar{A}_{\nu} = \bar{B}_u.$$

Therefore

$$(f/m)_{\nu} = (g/n)_u.$$

Hence there exist two functions $\theta_1(u, \nu)$, $\theta_2(u, \nu)$ such that

$$\theta_{1u} = A, \quad \theta_{1\nu} = B,$$

$$\theta_{2u} = f/m, \quad \theta_{2\nu} = g/n.$$

Consequently we remove from system (G) the coefficients A , B , f , g by a transformation of the form (12), where

$$\lambda = e^{-\theta_2}, \quad \mu = e^{\theta_1}.$$

It follows that the coefficients $\bar{A}$, $\bar{B}$, $\bar{f}$, $\bar{g}$, of system $(\bar{G})$ are also zero. If we assume that S_x and S_y are not radial transforms, then m is distinct from n , and from the next to the last integrability conditions of systems (G) and $(\bar{G})$ we have

$$c = \bar{c} = 0.$$

Therefore a non-singular congruence, whose reference surfaces S_x , S_y are in the relation of a fundamental transformation with respect to the congruence, is defined* by the following system, to be known as system (L):

$$(L) \quad \begin{aligned} x_{uu} &= p x + a x_u + \beta x_{\nu} + L y, \\ x_{u\nu} &= a x_u + b x_{\nu}, \\ x_{\nu\nu} &= q x + \gamma x_u + \delta x_{\nu} + N y, \\ y_u &= m x_u, \quad y_{\nu} = n x_{\nu}, \end{aligned}$$

*E. P. Lane, On the Fundamental Transformation of Surfaces. Annals of Mathematics, Second Series, Vol. 30 (1929), p. 460.

in which it is assumed that

$$m \neq n, \quad \Delta = mn \neq 0.$$

The focal points of a generator l_{xy} of a congruence defined by system (L) are

$$\xi = mx - y, \quad \eta = mx - y.$$

In order that the focal surfaces S_ξ , S_η do not degenerate into curves we shall assume that $m_u m_v \neq 0$.* The asymptotic curves on S_ξ and S_η are defined† respectively by the equations

$$(n-m)\beta du^2 + m_v dv^2 = 0, \\ m_u du^2 + (m-n)\gamma dv^2 = 0.$$

We shall henceforth suppose that $\beta\gamma \neq 0$, so that both focal surfaces will be non-developable.

The integrability conditions of system (L) are as follows:

$$\begin{aligned} a_u - a_v + ab - \beta\gamma &= 0, \\ b_u - \beta_v + a\beta + b(b-a) - \beta S - \rho &= Lm, \\ -\rho_v + a\rho &= -\beta q = 0, \\ -L_v + aL &= -\beta N = 0, \\ b_v - \delta_u + ab - \beta\gamma &= 0, \\ a_v - \gamma_u + b\gamma + a(a-S) - \gamma\delta - q &= Nm, \\ -q_u + bq &= -\gamma\rho = 0, \\ -N_u + bN &= -\gamma L = 0, \\ m_v &= -a(m-n) = 0, \\ m_u &= -b(m-n) = 0. \end{aligned}$$

*Lane, loc. cit., p. 468.

†Ibid., p. 469.

Corresponding to system $(\bar{G})$, we have the following system $(\bar{L})$:

$$\begin{aligned}
 (\bar{L}) \quad y_{uu} &= mLy + (\alpha + m_u/m)y_u + (m\beta/n)y_v + m\phi\alpha, \\
 y_{uv} &= (am/m)y_u + (bm/m)y_v, \\
 y_{vv} &= mNy + (m\gamma/m)y_u + (S + m_v/m)y_v + m\phi\gamma, \\
 x_u &= y_u/m, \quad x_v = y_v/m.
 \end{aligned}$$

It has been shown that the axis of P_x with respect to the parametric net of S_x , and the axis of P_y with respect to the parametric net of S_y , coincide with ℓ_{xy} if, and only if,

$\beta = \gamma = 0$, in which case the parametric curves of S_x and S_y are plane curves.* Hence the assumption $\beta\gamma \neq 0$ restricts Γ_{xy} from being the axis congruence of the parametric nets of S_x and S_y .

The most general point-transformation which leaves system (L) invariant is

$$x = k_1 \bar{x}, \quad y = k_2 \bar{y}, \quad \bar{u} = U(u), \quad \bar{v} = V(v),$$

in which k_1, k_2 are arbitrary constants, U and V are arbitrary functions of the variables indicated.

*Lane, loc. cit., p. 469.

5. SYSTEM (S)

Wilczynski has studied a one-parameter family of non-developable ruled surfaces by means of a completely integrable system (S) of four homogeneous linear partial differential equations with two dependent and two independent variables, two of the equations of the system being of the first order and two of the second order.* A second system (S'), which is symmetric with system (S) in the independent variables, has been obtained from the latter, and this second system defines a second one-parameter family of non-developable ruled surfaces except for a projective transformation. The two one-parameter families constitute a net of ruled surfaces in a congruence. The invariants of this net have been found by Wilczynski, and in terms of these the invariants of the one-parameter family of ruled surfaces defined by system (S) have been obtained; the latter are those invariants of the net which do not change when the second family of the net is changed arbitrarily. It is our purpose now to show that systems (G) and (S) are equivalent, in the sense that the one may be obtained from the other. We are thus enabled to make use of the theory developed by Wilczynski to find analytically a unique net of ruled surfaces in a congruence defined by system (G), for which a certain net invariant has the value zero; it will be later shown that the geometric relation between pairs of ruled surfaces, one from each family of this net, is such that the net is a conjugate net.

*Wilczynski, One-Parameter Families.

Let us assume that the parametric ruled surfaces R_u ($v = \text{const.}$) of a congruence defined by system (G) are non-developable. Then the coefficient s is not zero, and from the fourth equation of this system we determine x_v in terms of x_u, y_u, x , and y . If this value of x_v is substituted in the fifth equation of system (G) we obtain y_v in terms of x_u, y_u, x, y ; similarly, by substituting for x_v in the first equation of system (G), we obtain x_{uu} in terms of x_u, y_u, x, y . Finally, by differentiating the fourth equation of system (G) with respect to u and making use of the values previously obtained for x_v, x_{uu} and of x_{uv} from the second equation of system (G), we determine y_{uu} in terms of x_u, y_u, x, y . We thus derive from four equations of system (G) the following system (S):

$$(S) \quad \begin{aligned} x_{uu} + p_{11}x_u + p_{12}y_u + g_{11}x + g_{12}y &= 0, \\ y_{uu} + p_{21}x_u + p_{22}y_u + g_{21}x + g_{22}y &= 0, \\ x_v &= a_{11}x_u + a_{12}y_u + b_{11}x + b_{12}y, \\ y_v &= a_{21}x_u + a_{22}y_u + b_{21}x + b_{22}y, \end{aligned}$$

in which

$$\begin{aligned} p_{11} &= (m\beta - sd)/s, & g_{11} &= (f\beta - sp)/s, \\ p_{12} &= -\beta/s, & g_{12} &= (A\beta - sL)/s, \\ p_{21} &= \Delta\bar{\beta}/s = (mg_{12} - sg_{11})/s, \\ p_{22} &= -(n\bar{\beta} + s\bar{d})/s = -(sA + g_{12})/s, \\ g_{21} &= -(\Delta\bar{A}\bar{\beta} + s\bar{L})/s = -(sg_{13} - fg_{12})/s, \\ g_{22} &= -(\Delta\bar{f}\bar{\beta} + s\bar{p})/s = -\{s(A_u + g_{24}) - Ag_{12}\}/s, \end{aligned}$$

$$\begin{aligned}a_{11} &= -m/s, & b_{11} &= -f/s, \\a_{12} &= 1/s, & b_{12} &= -A/s, \\a_{21} &= -\Delta/s, & b_{21} &= \Delta \bar{A}/s, \\a_{22} &= n/s, & b_{22} &= \Delta \bar{f}/s,\end{aligned}$$

where $\bar{p}$, $\bar{\alpha}$, $\bar{\beta}$, $\bar{L}$, $\bar{f}$, $\bar{A}$ are given by equations (1) and (2). Every completely integrable system of the form (S) defines a one-parameter family of non-developable parametric ruled surfaces R_u , except for a projective transformation.

If we assume that

$$a_{11}a_{22} - a_{12}a_{21} \neq 0,$$

which is equivalent to assuming that t is not zero in system (G), a second system (S') can be obtained from system (S), namely:

$$\begin{aligned}(S') \quad & x_{vv} + r_{11}x_v + r_{12}y_v + s_{11}x + s_{12}y = 0, \\& y_{vv} + r_{21}x_v + r_{22}y_v + s_{21}x + s_{22}y = 0, \\& x_u = g_{11}x_v + g_{12}y_v + h_{11}x + h_{12}y, \\& y_u = g_{21}x_v + g_{22}y_v + h_{21}x + h_{22}y.\end{aligned}$$

The coefficients r_{ij} , s_{ij} , g_{ij} , h_{ij} are defined in terms of the coefficients of system (S).^{*} System (S') defines a second one-parameter family of non-developable parametric ruled surfaces R_v ($u = \text{const.}$) except for a projective transformation.

Conversely, we obtain system (G) from system (S), as follows. From the last two equations of system (S), the following expressions are found by differentiation and making use of the first two equations of that system:

^{*}Wilczynski, One-Parameter Families, p. 163.

$$(18) \quad \begin{aligned} x_{uv} &= c_{11}x_u + c_{12}y_u + d_{11}x + d_{12}y, \\ y_{uv} &= c_{21}x_u + c_{22}y_u + d_{21}x + d_{22}y, \\ x_{vv} &= e_{11}x_u + e_{12}y_u + f_{11}x + f_{12}y, \end{aligned}$$

where c_{ij} , d_{ij} , e_{ij} , f_{ij} , ($i, j = 1, 2$), are defined in terms of the coefficients of system (S).* Let us assume that a_{12} is not zero; then the surface S_x is not a focal surface of a congruence defined by system (S). From the third equation of system (S) we determine y_u in terms of x , x_u , x_v , and y . By substituting this value of y_u in the first and fourth equations of system (S) we obtain x_{uu} and y_v , each in terms of x , x_u , x_v , y . Finally, by substituting for y_u in the first and third of equations (18) we obtain the second and third equations of a system of differential equations of the form (G). The coefficients of system (G), in terms of the coefficients of system (S), are as follows:

$$\begin{aligned} p &= (b_{11}f_{12} - a_{12}g_{11})/a_{12}, & q &= (a_{12}f_{11} - b_{11}e_{12})/a_{12}, \\ \alpha &= (a_{11}f_{12} - a_{12}f_{11})/a_{12}, & \gamma &= (a_{12}e_{11} - a_{11}e_{12})/a_{12}, \\ \beta &= -f_{12}/a_{12}, & \delta &= e_{12}/a_{12}, \\ L &= (b_{12}f_{12} - a_{12}g_{12})/a_{12}, & N &= (a_{12}f_{12} - b_{12}e_{12})/a_{12}, \\ a &= (a_{12}c_{11} - a_{11}c_{12})/a_{12}, & b &= c_{12}/a_{12}, \\ c &= (a_{12}d_{11} - b_{11}c_{12})/a_{12}, & M &= (a_{12}d_{12} - b_{12}c_{12})/a_{12}, \\ f &= -b_{11}/a_{12}, & g &= (a_{12}b_{21} - a_{22}b_{11})/a_{12}, \\ m &= -a_{11}/a_{12}, & t &= (a_{12}a_{21} - a_{11}a_{22})/a_{12}. \end{aligned}$$

*Wilczynski, One-Parameter Families, p. 163.

$$s = 1/a_{12} \quad , \quad m = a_{22}/a_{12} \quad ,$$

$$A = -b_{12}/a_{12} \quad , \quad B = (a_{12}b_{22} - a_{22}b_{12})/a_{12}.$$

The parametric ruled surfaces, R_u and R_v , of the congruence Γ_{xy} are not altered by applying to the defining system (G) a transformation of the form

$$(19) \quad \bar{u} = U(u) \quad , \quad \bar{v} = V(v) \quad ,$$

where U and V are arbitrary functions of u and v respectively. Those functions of seminvariants of system (G) which are not changed under any transformation of the form (19), except possibly for a factor involving only the derivatives U' and V' , are called invariants of the parametric net of ruled surfaces, or, more shortly, net invariants, it being understood that the parametric net is under consideration.* It may be readily verified that the functions

$$(n-m)/s \quad , \quad t/s \quad , \quad \Omega/s \quad ,$$

and the corresponding three seminvariants obtained by interchanging s and t , are all relative net invariants.

Let us apply to system (G) a particular transformation of the form

$$(20) \quad \bar{u} = \phi(u, v) \quad , \quad \bar{v} = V(v) \quad ,$$

where V is an arbitrary function of the single variable v , and ϕ is a solution (different from $\phi = \text{const.}$) of the partial differential equation

$$(21) \quad 2\phi_v - \phi_u(n-m)/s = 0.$$

Denoting the coefficients of the transformed system by dashes,

*Wilczynski, One-Parameter Families, p. 173.

we find

$$\begin{aligned}\bar{m} &= (m+n)/2, \quad \bar{s} = V'\alpha/\phi_u, \\ \bar{n} &= (m+n)/2, \quad \bar{t} = \phi_u \Omega^2 / 4V'\alpha.\end{aligned}$$

Therefore

$$\bar{m} - \bar{n} = 0,$$

and $\bar{t}$ is different from zero. Since the most general solution of (21) is an arbitrary function of $\bar{u}$, the transformation (20) associates with the one-parameter family of non-developable ruled surfaces R_u a uniquely determined one-parameter family of non-developable ruled surfaces $R_{\bar{u}}$, the two families constituting a net which is characterized analytically by the vanishing of the net invariant $(n-m)/\alpha$. Wilczynski has called such a net a conjugate net of ruled surfaces.* Referred to the original parameters (u,v) , this conjugate net of ruled surfaces is defined by the ordinary differential equation

$$(22) \quad 2du dv + \{(n-m)/\alpha\} dv^2 = 0.$$

Since the coefficients of (22) are absolute seminvariants of system (G), this equation determines the curves of intersection of the ruled surfaces of the conjugate net with any transversal surface of the congruence. Hence if we impose on system (G) the conditions

$$\Omega \neq 0, \quad n = m,$$

then the focal surfaces of the congruence Γ_{xy} are distinct, and the parametric ruled surfaces are non-developable and form a conjugate net, in the sense of Wilczynski.

*Wilczynski, One-Parameter Families, p. 180.

6. NECESSARY AND SUFFICIENT CONDITIONS THAT A NET OF RULED SURFACES BE CONJUGATE

Let S be any transversal surface of a congruence Γ_{xy} . A net $\mathcal{N}$ of ruled surfaces in Γ_{xy} intersects S in the curves of a net, and likewise the developables of Γ_{xy} trace out a net of curves on S . The surface S shall be said to possess property $\mathcal{L}$ with respect to the net $\mathcal{N}$ if the ruled surfaces of $\mathcal{N}$ intersect S in curves whose tangents at any point separate harmonically the tangents at the point to the curves which the developables trace out on S . The definition of a conjugate net may then be re-stated as follows: a net $\mathcal{N}$ of ruled surfaces in a congruence is a conjugate net if every transversal surface of the congruence possesses property $\mathcal{L}$ with respect to $\mathcal{N}$. It is the aim of this section to show that if the developables of Γ_{xy} are known, and if any transversal surface S of Γ_{xy} is given, then a net $\mathcal{N}$ of ruled surfaces in Γ_{xy} is a conjugate net if S possesses property $\mathcal{L}$ with respect to $\mathcal{N}$.

Let us assume that the parametric ruled surfaces, R_u and R_v , of the general congruence Γ_{xy} are not developable; then the coefficients s and t are both different from zero in system (G). The developables of Γ_{xy} are defined by

$$(23) \quad s du^2 + (n-m) du dv - t dv^2 = 0,$$

whose discriminant, Ω^2 , is by a previous assumption different from zero. Unless otherwise specified, we shall employ the tetrahedron of reference whose vertices are the points x , x_u , x_v , and y . With respect to this tetrahedron and a suitably

chosen unit point, an expression of the form

$$x_1 x + x_2 x_u + x_3 x_v + x_4 y$$

represents a point whose local coordinates are (x_1, x_2, x_3, x_4) .

The equation of the pair of planes which are tangent to R_u and R_v at a point $P = x + \omega y$ is

$$(24) \quad \omega(1+\omega n)x_1^2 - \{ \omega^2 + (1+\omega m)(1+\omega m) \} x_1 x_3 + \omega(1+\omega m)x_3^2 = 0,$$

and the focal planes of ℓ_{xy} are given by

$$(25) \quad \omega x_1^2 + (n-m)x_1 x_3 - \omega x_3^2 = 0.$$

The pair of planes (24) is divided harmonically by the focal planes (25) if, and only if,

$$(n-m)E(\omega) = 0.$$

The factor $E(\omega)$ vanishes only at a focal point of ℓ_{xy} , at which place the tangent planes of R_u and R_v coincide with a focal plane. Hence the planes (24) are separated harmonically by the focal planes (25) if, and only if,

$$(26) \quad n = m.$$

The condition (26) is independent of ω . It follows that if one point of ℓ_{xy} can be found, such that the tangent planes to R_u and R_v at this point are separated harmonically by the focal planes of ℓ_{xy} , then every point of ℓ_{xy} possesses this same property; in other words, the parametric net is conjugate.

Dual considerations lead to a similar conclusion. Any plane which contains ℓ_{xy} and is not a focal plane of ℓ_{xy} is given by

$$(27) \quad x_2 - k x_3 = 0,$$

where

$$(28) \quad \omega k^2 + (n-m)k - \omega \neq 0.$$

The plane (27) is tangent to R_u and R_v at the points $x + \omega_i y$, ($i = 1, 2$), where

$$\omega_1 = 1/(\kappa\Delta - m) \quad , \quad \omega_2 = \kappa/(t - \kappa m).$$

The focal points of ℓ_{xy} are given by $x + \omega_j y$, ($j = 3, 4$), where

$E(\omega_j) = 0$. The points of tangency with R_u and R_v , of the plane (27), are separated harmonically by the focal points of ℓ_{xy} if, and only if,

$$(29) \quad (\omega_1 \ \omega_2 \ \omega_3 \ \omega_4) = -1.$$

The condition (29) reduces to

$$(m-m)\{\Delta \kappa^2 + (m-m)\kappa - t\} = 0,$$

which, in view of (28), is equivalent to

$$n = m.$$

Hence if one plane (27) has the property that the points of tangency of this plane with R_u and R_v separate harmonically the focal points of ℓ_{xy} , then every plane through ℓ_{xy} has this same property, and the parametric net of ruled surfaces is consequently a conjugate net.

We have seen that the ratios of the coefficients of (23) are absolute seminvariants of system (G). The condition $n = m$ is a necessary and sufficient condition that every transversal surface of Γ_{xy} possess property $\mathcal{L}$ with respect to the parametric net of ruled surfaces of Γ_{xy} . If one transversal surface possesses property $\mathcal{L}$ with respect to the parametric net of Γ_{xy} , then every transversal surface of Γ_{xy} possesses property $\mathcal{L}$ with respect to this net, and therefore the parametric net of ruled surfaces is conjugate.

Let corresponding points, P_x and P_y , trace out the curves defined on the reference surfaces by the quadratic differential equation

$$(30) \quad P du^2 + 2Q du dv + R dv^2 = 0,$$

where P, Q, R are functions of u and v such that $Q^2 - PR \neq 0$.

The line ℓ_{xy} will then generate a net $\mathcal{N}$ of ruled surfaces in the congruence Γ_{xy} . Let $\lambda_i = (dv/du)_i$, ($i = 1, 2$), be the directions defined by (30) at any point of a reference surface, and let us assume that λ_i do not satisfy (23). The resultant of (23) and (30) is therefore different from zero for all values of u and v ; that is

$$(31) \quad \{R\Delta - P\Gamma - Q(n-m)\}^2 - \Omega^2(Q^2 - PR) \neq 0.$$

The ruled surfaces R_1 and R_2 , corresponding to the curves whose directions at any point of a reference surface are given by λ_1 and λ_2 , are then non-developable. The equations of the planes tangent to R_i at any point $x + \omega y$ are given by

$$(\omega\Delta + \lambda_i + \omega m \lambda_i) x_2 - (1 + \omega m + \omega \Gamma \lambda_i) x_3 = 0 \quad (i=1, 2),$$

and these planes are divided harmonically by the focal planes (25) if, and only if,

$$\{2\Delta - 2\Gamma \lambda_1 \lambda_2 + (n-m)(\lambda_1 + \lambda_2)\} E(\omega) = 0,$$

that is, if and only if

$$(32) \quad R\Delta - P\Gamma - Q(n-m) = 0.$$

The condition (32) is independent of ω , and is a necessary and sufficient condition that every transversal surface of Γ_{xy} possess property $\mathcal{L}$ with respect to the net $\mathcal{N}$.

Theorem: If any transversal surface of a congruence possesses property $\mathcal{L}$ with respect to a net $\mathcal{N}$ of ruled surfaces in the congruence, then $\mathcal{N}$ is conjugate.

The jacobian J of the quadratic forms defined by the left members of (23) and (30) is given by

$$2J = \{2Q\Delta - P(m-m)\}du^2 + 2(R\Delta + PT)du dv + \{2QT - R(m-m)\}dv^2.$$

Since the discriminant of this jacobian, apart from a numerical factor, is given by (31), then $J = 0$ defines a net of curves on a reference surface of Γ_{xy} ; the curves of this net separate harmonically the curves of the nets defined by (23) and (30). Hence, when (30) is given, a unique conjugate net of ruled surfaces is determined by

$$(33) \quad \{2Q\Delta - P(m-m)\}du^2 + 2(R\Delta + PT)du dv + \{2QT - R(m-m)\}dv^2 = 0.$$

If (32) is satisfied, then (30) and (33) determine associate conjugate nets of ruled surfaces in the congruence.

The ruled surface R_v of Γ_{xy} is touched in the point $\xi = m\chi - y$ by the plane of ℓ_{xy} and the corresponding tangent to the curve $v = \text{const.}$ on S_x . Likewise the ruled surface R_u is touched in the point $\eta = m\chi - y$ by the plane of ℓ_{xy} and the corresponding tangent to the curve $u = \text{const.}$ on S_x . The harmonic conjugate of the point x with respect to ξ and η is given by the point

$$\zeta = (m+m)\chi - 2y.$$

The focal points of ℓ_{xy} are given by $x + \omega y$, where

$$\omega = \{-(m+m) \pm \Omega\} / 2\Delta.$$

It was observed by Green that ζ coincides with the harmonic

conjugate of the point x with respect to the focal points of ℓ_{xy} . If the parametric net of ruled surfaces is conjugate, then the focal points of ℓ_{xy} are

$$z_1 = (mx - y) - \sqrt{st} \alpha, \quad z_2 = (mx - y) + \sqrt{st} \alpha.$$

In this case, the points ξ , η , and ζ coincide, and the points z_1 , z_2 separate harmonically the points x and $mx - y$. Further, if $n = m = 0$, the surfaces S_x and S_y are point-conjugates with respect to the congruence which contains the conjugate parametric net of ruled surfaces.

If the parametric ruled surfaces of the congruence Γ_{xy} intersect S_x in the asymptotic curves on the latter, then Γ_{xy} is defined by system (F). For this congruence a necessary and sufficient condition that the parametric net of ruled surfaces be conjugate is

$$n = m.$$

which is equivalent to

$$a_u = b_v.$$

The tangents at any point x to the curves of intersection of S_x with the developables and the asymptotic tangents at that point determine an involution, the double rays of which are defined by

$$(34) \quad s du^2 + t dv^2 = 0.$$

Hence the net of ruled surfaces determined by (34) is a conjugate net and intersects S_x in a conjugate net of curves. The associate conjugate net of ruled surfaces is given by

$$s(m - m) du^2 - 4st du dv + t(m - n) dv^2 = 0,$$

which, if $m = m$, reduces to the parametric net.

Let the congruence Γ_{xy} be defined by the second canonical form, system (L). The net of ruled surfaces determined by (30) is composed of non-developables if $PR \neq 0$. A necessary and sufficient condition that the net of ruled surfaces determined by (30) be conjugate is given by $Q = 0$.

Theorem: Let Γ_{xy} be the conjugate congruence of a fundamental transformation. Every equation of the form

$$P(u,v)du^2 + R(u,v)dv^2 = 0 \quad (PR \neq 0),$$

determines a conjugate net of ruled surfaces in Γ_{xy} . The associate conjugate net of ruled surfaces is given by

$$P(u,v)du^2 - R(u,v)dv^2 = 0.$$

The double rays of the involution determined by the asymptotic and parametric tangents of S_x at any point x are given by

$$(35) \quad L du^2 - N dv^2 = 0.$$

Consequently (35) determines a conjugate net of ruled surfaces in Γ_{xy} which intersects S_x in a conjugate net of curves.

7. THE QUADRIC SURFACES WHICH OSCULATE A PAIR OF RULED SURFACES OF A NET

Any pair of ruled surfaces, R_1 and R_2 , one from each family of a net $\mathcal{N}$, has a generator ℓ in common. The quadric surface which osculates R_1 along ℓ will be denoted by Q_1 , and similarly Q_2 will represent the quadric which osculates R_2 along ℓ . There exist, then, in association with a net $\mathcal{N}$, two systems of ∞^2 osculating quadrics. We wish to show in this section that the conjugacy of $\mathcal{N}$ is equivalent to the apolarity of any pair of quadrics, Q_1 and Q_2 , one from each of the two systems in association with $\mathcal{N}$.

Let us consider first the parametric net of the general congruence Γ_{xy} . The quadric surfaces, Q_u and Q_v , which osculate respectively the ruled surfaces R_u and R_v along ℓ_{xy} are given by

$$(36) \quad \beta x_1^2 + a_{33} x_3^2 - 2s x_1 x_3 + 2a_{23} x_2 x_3 + 2s^2 x_1 x_4 - 2sm x_3 x_4 = 0,$$

and

$$(37) \quad b_{22} x_2^2 + t y x_3^2 - 2t x_1 x_2 + 2b_{23} x_2 x_3 - 2tm x_2 x_4 + 2t^2 x_3 x_4 = 0,$$

where

$$a_{33} = f + mA - m_u - sa, \quad a_{23} = (s_u + sb - sA - sa)/2,$$

$$b_{22} = g + mB - m_v - tb, \quad b_{23} = (t_v + ta - tB - ts)/2.$$

The discriminants of Q_u and Q_v are respectively s^6 and t^6 , hence the quadrics are non-singular. The discriminant of any quadric of the pencil $Q_v - \kappa Q_u$ is given by

$$D(\kappa) = \Theta_0 - \Theta_1 \kappa + \Theta_2 \kappa^2 - \Theta_3 \kappa^3 + \Theta_4 \kappa^4,$$

where

$$\Theta_0 = t^6, \quad \Theta_4 = s^6,$$

$$\begin{aligned}\theta_1 &= 2st^4(n-m) \quad , \quad \theta_3 = 2s^4t(m-n) \quad , \\ \theta_2 &= s^2t^2\{(n-m)^2 - 2st\}.\end{aligned}$$

Since Q_u and Q_v have a generator in common, then $D(\kappa) = 0$ has two distinct double roots,* which are given by the equation

$$s^3\kappa^2 + st(n-m)\kappa - t^3 = 0,$$

whose discriminant is $(st\Omega)^2$.

The equations of Q_u and Q_v in plane coordinates u_i are respectively

$$(38) \quad \alpha_{11}u_1^2 - s^3\beta u_4^2 - 2s^4mu_1u_2 - 2s^5u_1u_3 + 2\alpha_{14}u_1u_4 + 2s^4u_2u_4 = 0,$$

and

$$(39) \quad \beta_{11}u_1^2 - t^3\gamma u_4^2 - 2t^5u_1u_2 - 2t^4nu_1u_3 + 2\beta_{14}u_1u_4 + 2t^4u_2u_4 = 0,$$

where α_{ij} is the cofactor of a_{ij} in the discriminant of (36), and β_{ij} is the cofactor of b_{ij} in the discriminant of (37).

The polar of the quadratic form (36) with respect to the quadratic form (37) is obtained by replacing the variable u_i in the left member of (38) by the differential operator $\partial/\partial x_i$ and then operating on the left member of (37).† We find that the polar of (36) with respect to (37) is given by $2\theta_3$, and the polar of (37) with respect to (36) is given by $2\theta_1$. The quantities θ_1 and θ_3 are projective invariants whose vanishing characterizes a pair of apolar forms.* Hence the quadrics

*G. Salmon, Analytic Geometry of Three Dimensions, Sixth Edition, Vol. I, Longmans, Green, & Co., (1914), p. 205.

†R. M. Winger, Projective Geometry, D. C. Heath & Co., (1923), p. 282.

*Winger, loc. cit., p. 283. See also Snyder and Sisam, Analytic Geometry of Space, Henry Holt & Co., (1914), pp. 148, 181.

Q_u and Q_v are apolar quadrics if, and only if, the parametric net of ruled surfaces is conjugate.

Let R_1 and R_2 be a pair of ruled surfaces, one from each family of the net determined by (30). It has been previously assumed that the directions $\lambda_i = (dv/du)_i$, ($i=1,2$), which are defined by (30) at any point of a reference surface of Γ_{xy} , do not satisfy (23), so that R_1 and R_2 are non-developable. The quadrics Q_1 and Q_2 , which osculate R_1 and R_2 along ℓ_{xy} , are given by

$$A_{22}x_2^2 + A_{33}x_3^2 - 2\lambda_1 x_1 x_2 + 2\lambda_1 x_1 x_3 + 2A_{23}x_2 x_3 - 2(\Delta + \lambda_1 m)x_2 x_4 + 2(m + \lambda_1 t)x_3 x_4 = 0,$$

and

$$B_{22}x_2^2 + B_{33}x_3^2 - 2\lambda_2 x_1 x_2 + 2\lambda_2 x_1 x_3 + 2B_{23}x_2 x_3 - 2(\Delta + \lambda_2 m)x_2 x_4 + 2(m + \lambda_2 t)x_3 x_4 = 0,$$

where the coefficients A_{22} , A_{23} , A_{33} , B_{22} , B_{23} , B_{33} are complicated expressions whose values will not be needed. The discriminant of Q_1 is given by D_1^2 , where

$$D_i = \Delta + (m-m)\lambda_i - t\lambda_i^2 \quad (i=1,2),$$

hence Q_1 is non-singular. The discriminant of any quadric of the pencil $Q_2 - \kappa Q_1$ is given by

$$[D_1 \kappa^2 - \{2\Delta - 2t\lambda_1 \lambda_2 + (m-m)(\lambda_1 + \lambda_2)\}\kappa + D_2]^2.$$

Thus the quadrics Q_1 and Q_2 are apolar if, and only if

$$2\Delta - 2t\lambda_1 \lambda_2 + (m-m)(\lambda_1 + \lambda_2) = 0,$$

that is, if and only if (32) is satisfied.

Theorem: The quadric surfaces which osculate a pair of ruled surfaces of a net are apolar quadrics if, and only if, the net is a conjugate net.

8. CONJUGATE NETS OF RULED SURFACES IN A PAIR OF GREEN-RECIPROCAL CONGRUENCES

In this section an arbitrary pair of Green-reciprocal congruences, Γ_1 and Γ_2 , are considered. To any net $\mathcal{N}_1$ of ruled surfaces in Γ_1 there corresponds, in the Green-reciprocal relation, a net $\mathcal{N}_2$ of ruled surfaces in Γ_2 . The directrix congruences of Wilczynski are the only reciprocal congruences which possess the property that the conjugacy of either $\mathcal{N}_1$ or $\mathcal{N}_2$ implies the conjugacy of the other; this follows directly from the fact that the developables of the directrix congruences correspond to the same net of curves on a reference surface. There exists a unique pair of congruences, namely, the projective normal congruence of a surface and the reciprocal of the projective normal congruence, each of which contains a conjugate net of ruled surfaces.

The analytic representation of a pair of Green-reciprocal congruences is well-known. Let Γ_1 be the congruence of lines ℓ_1 joining the points P_x and P_y and defined by system (F). The Γ_1 -curves of S_x are given by (23), which may be rewritten in the form

$$(40) \quad (F - 2a\beta + \beta\psi_2)du^2 + (a_u - b_v)du dv - (G - 2b\gamma + \gamma\psi_1)dv^2 = 0,$$

where the functions ψ_1 and ψ_2 are defined in §4, and

$$F = f - b_u + b\theta_u - b^2 + a\beta,$$

$$G = g - a_v + a\theta_v - a^2 + b\gamma.$$

The equation of the quadric of Lie which osculates S_x at P_x is found to be

$$(41) \quad \{ab + c - (\beta\gamma + \theta_{uv})/2\}x_4^2 - x_1x_4 + x_2x_3 - b x_2x_4 - a x_3x_4 = 0.$$

Let ℓ_2 be the line which joins the points

$$\rho = x_u - b x_v, \quad \sigma = x_v - a x_u.$$

The tangent planes to the quadric (41) at the points ρ and σ are respectively $x_3 = 0$ and $x_2 = 0$. Therefore the lines ℓ_1 and ℓ_2 are reciprocal polars with respect to the quadric of Lie. As ℓ_1 generates the congruence Γ_1 , the corresponding line ℓ_2 generates a congruence Γ_2 ; the two congruences, Γ_1 and Γ_2 , constitute a pair of reciprocal congruences, in the sense of Green. The Γ_2 -curves of S_x are defined by

$$(42) \quad F du^2 + (a_u - b_v) du dv - G dv^2 = 0,$$

and the focal points of ℓ_2 are given by $\rho + \omega \sigma$, where

$$(43) \quad G \omega^2 + (a_u - b_v) \omega - F = 0.$$

To the parametric ruled surfaces R_u and R_v of Γ_1 there correspond the parametric ruled surfaces $R^{(u)}$ and $R^{(v)}$ of Γ_2 ; we shall assume that both F and G are different from zero, so that $R^{(u)}$ and $R^{(v)}$ will be non-developable. It may be demonstrated, by either of the methods described in §6, that the parametric net of ruled surfaces of Γ_2 is conjugate if, and only if

$$(44) \quad a_u = b_v,$$

which is equivalent to (26). Hence the parametric net of Γ_2 is conjugate if, and only if, the parametric net of Γ_1 is conjugate; in this case, Γ_2 is harmonic to S_x and Γ_1 is conjugate to S_x .

In the presence of (44), equation (43) is the same as equation (42), the latter being considered as a quadratic in

dv/du . It follows that if the parametric nets of Γ_1 and Γ_2 are conjugate, the tangents to the Γ_2 -curves at any point of S_X intersect the corresponding line ℓ_2 in the focal points of ℓ_2 ; this, of course, is a consequence of the fact that Γ_2 is harmonic to S_X . Since the condition for conjugate parametric nets of Γ_1, Γ_2 is identical with the condition that the congruences be respectively conjugate and harmonic to S_X , further geometric criteria for the latter situation, as developed by Green,* may be applied to the former situation.

Let

$$h = ab - a_u + c, \quad k = ab - b_v + c.$$

Then

$$C_v = hx + y + q\rho, \quad \sigma_u = hx + y + b\sigma.$$

Therefore the line ℓ_1 is intersected in the point $\xi' = hx + y$ by the corresponding tangent to the curve $u = \text{const.}$ on the surface S_ρ . Similarly, the line ℓ_2 is intersected in the point $\eta' = kx + y$ by the corresponding tangent to the curve $v = \text{const.}$ on the surface S_σ . Green has shown that the line ℓ_1 intersects the quadric (41) in the double points of the involution determined by the pairs of points ξ, ξ' and η, η' ,† where the points

$$\xi = mx - y, \quad \eta = mx - y$$

are defined geometrically in §6. Evidently a necessary and sufficient condition that the parametric nets of Γ_1 and Γ_2 be conjugate is that ξ coincides with η and hence ξ' coincides

*Green, Surfaces, pp. 101-102.

†Green, loc. cit., p. 98.

with η' .

As the line ℓ_1 generates the net $\mathcal{N}_1$ determined by (30), the corresponding line ℓ_2 generates a net $\mathcal{N}_2$ in Γ_2 ; the ruled surfaces of the latter net will be non-developable provided the directions $\lambda_i = (dv/du)_i$, $(i=1,2)$, which are defined by (30) at any point of S_X , do not satisfy (42). It will therefore be assumed that

$$\{RF - PG - Q(a_u - b_v)\}^2 - \{(a_u - b_v)^2 + 4FG\}\{Q^2 - PR\} \neq 0.$$

Let $R^{(1)}$ and $R^{(2)}$ represent ruled surfaces of $\mathcal{N}_2$ corresponding respectively to curves whose directions at any point of S_X are given by λ_1 and λ_2 . Any plane which contains ℓ_2 is given by

$$(45) \quad x_1 + b x_2 + a x_3 - \kappa x_4 = 0,$$

where κ is a parameter. The plane (45) is tangent to $R^{(1)}$ at the point $\rho + \omega_i \sigma$, where

$$\omega_i = \{F - \lambda_i(\kappa - h)\} / (\kappa - h - G\lambda_i) \quad (i=1,2).$$

The focal points of ℓ_2 are given by $\rho + \omega_j \sigma$, $(j=3,4)$, where ω_j is a root of (43). Therefore the net $\mathcal{N}_2$ is conjugate if, and only if, the cross-ratio

$$(\omega_1, \omega_2, \omega_3, \omega_4)$$

is equal to negative unity for all values of κ . We thus find that $\mathcal{N}_2$ is conjugate if, and only if,

$$(46) \quad RF - PG - Q(a_u - b_v) = 0.$$

Hence a necessary and sufficient condition that any net $\mathcal{N}_2$ in Γ_2 be conjugate is that the curves on S_X which correspond to the ruled surfaces of $\mathcal{N}_2$ separate harmonically the curves on S_X which correspond to the developables of Γ_2 .

Since

$$s = F - 2a\beta + \beta\psi_2,$$

$$t = G - 2b\gamma + \gamma\psi_1,$$

and

$$n - m = a_u - b_v,$$

then

$$R\Delta - Pt - Q(n - m) = RF - PG - Q(a_u - b_v) + R\beta(\psi_2 - 2a) - P\gamma(\psi_1 - 2b).$$

It follows that (32) and (46) are satisfied simultaneously if

$$a = \psi_2/2, \quad b = \psi_1/2.$$

In this case the Γ_1 -curves and Γ_2 -curves of S_X coincide, and Γ_1 and Γ_2 are the directrix congruences of Wilczynski. Hence the directrix congruences of Wilczynski are the only reciprocal congruences which possess the property that if any net of ruled surfaces in one congruence is conjugate then the corresponding net of ruled surfaces in the other congruence is conjugate.

It has been noted in §4 that the line ℓ_1 coincides with the projective normal through P_X of the surface S_X if

$$a = b = 0.$$

It follows from (44) that the parametric net of the congruence of projective normals of S_X is conjugate, and likewise the parametric net of the reciprocal of the projective normal congruence is conjugate. Hence if a non-ruled surface S is given, a covariant conjugate net of ruled surfaces is generated by the projective normal through a point P of S , as P traces out the asymptotic curves on S ; moreover, the Green-reciprocal of this net is also conjugate.

9. PAIRS OF CONGRUENCES WITH GENERATORS IN ONE-TO-ONE CORRESPONDENCE

A pair of congruences, Γ_1 and Γ_2 , whose generators, ℓ_1 and ℓ_2 , are skew lines in one-to-one correspondence, is defined except for a projective transformation by a completely integrable system of differential equations of the following form,* to be known as system (C):

$$(C) \quad x_u^i = a^{ij} x_j^i, \quad x_v^i = b^{ij} x_j^i \quad (i, j = 1, \dots, 4),$$

in which each x^i stands for $x_k^i = x_k^i(u, v)$, ($k = 1, \dots, 4$).

The line ℓ_1 is the join of the points P_1, P_2 , while the line ℓ_2 joins the points P_3 and P_4 , where P_i is a point whose homogeneous coordinates are given by x_k^i . The coefficients of the system satisfy the integrability conditions

$$a_v^{ij} + a^{ik} b^{kj} = b_u^{ij} + b^{ik} a^{kj}.$$

A canonical form exists for system (C) but is not essential for our purposes. We shall obtain, in determinantal form, conditions which are necessary and sufficient that the parametric ruled surfaces form a conjugate net in each of Γ_1 and Γ_2 ; these conditions reduce to the corresponding conditions of §8 when Γ_1 and Γ_2 are an arbitrary pair of Green-reciprocal congruences. In addition, two new properties of the directrix congruences of Wilczynski are developed.

Since the determinant

$$(x^1, x^2, x^3, x^4)$$

*A. J. Cook, Pairs of Rectilinear Congruences with Generators in One-to-One Correspondence, Transactions of the American Mathematical Society, Vol. 32 (1930), pp. 31-46.

is different from zero, we may use the four points P_i as the vertices of a local tetrahedron of reference. With respect to this tetrahedron and a suitably chosen unit point, an expression of the form

$$x_1 x^1 + x_2 x^2 + x_3 x^3 + x_4 x^4$$

represents a point whose local coordinates are x_1, x_2, x_3, x_4 .

For convenience, the following determinants are defined:

$$A_{ij}^{rs} = \begin{vmatrix} a^{ir} & a^{is} \\ a^{jr} & a^{js} \end{vmatrix}, \quad B_{ij}^{rs} = \begin{vmatrix} b^{ir} & b^{is} \\ b^{jr} & b^{js} \end{vmatrix},$$

$$\Delta_{ij}^{rs} = \begin{vmatrix} a^{ir} & a^{is} \\ b^{jr} & b^{js} \end{vmatrix}, \quad \nabla_{ij}^{rs} = \begin{vmatrix} a^{ir} & b^{is} \\ a^{jr} & b^{js} \end{vmatrix},$$

where $i, j, r, s = 1, \dots, 4$. Let us consider first the congruence Γ_1 . The developables of this congruence are determined by the differential equation

$$(47) \quad A_{12}^{34} du^2 + (\Delta_{12}^{34} - \Delta_{21}^{34}) du dv + B_{12}^{34} dv^2 = 0,$$

in which we shall assume that A_{12}^{34} and B_{12}^{34} are different from zero, so that the parametric ruled surfaces R_{1u}, R_{1v} of Γ_1 will be non-developable. The focal points of a generator ℓ_1 are given by $x^1 + \omega x^2$, where

$$(48) \quad \Delta_{22}^{34} \omega^2 + (\Delta_{12}^{34} + \Delta_{21}^{34}) \omega + \Delta_{11}^{34} = 0.$$

Any plane

$$x_3 - k x_4 = 0$$

is tangent to R_{1u} and R_{1v} at the points

$$x^1 + \omega_1 x^2, \quad x^1 + \omega_2 x^2,$$

respectively, where

$$\omega_1 = (\kappa a^{14} - a^{13}) / (a^{23} - \kappa a^{24}) \quad , \quad \omega_2 = (\kappa b^{14} - b^{13}) / (b^{23} - \kappa b^{24}) .$$

Hence the parametric net of ruled surfaces of Γ_1 is conjugate if, and only if, for all values of κ the cross ratio

$$(\omega_1 \ \omega_2 \ \omega_3 \ \omega_4)$$

is equal to negative unity, where ω_3, ω_4 are the roots of (48). This condition reduces to an identity in the parameter κ , viz.:

$$(49) \quad a_1 \kappa^2 + b_1 \kappa + c_1 \equiv 0 \ ,$$

where

$$a_1 = \nabla_{12}^{44} \{ \Delta_{21}^{34} - \Delta_{12}^{34} \} / \Delta_{22}^{34} \ ,$$

$$b_1 = -\{ (\nabla_{12}^{43})^2 - (\nabla_{12}^{34})^2 \} / \Delta_{22}^{34} = -\{ \nabla_{12}^{43} + \nabla_{12}^{34} \} \{ \Delta_{21}^{34} - \Delta_{12}^{34} \} / \Delta_{22}^{34} \ ,$$

$$c_1 = \nabla_{12}^{33} \{ \Delta_{21}^{34} - \Delta_{12}^{34} \} / \Delta_{22}^{34} \ .$$

Hence a necessary and sufficient condition that the parametric ruled surfaces of Γ_1 form a conjugate net is given by

$$(50) \quad a_1 = b_1 = c_1 = 0 \ .$$

Before writing a more explicit condition in the place of (50), we wish to show that a conjugate parametric net of ruled surfaces of Γ_1 is not defined in the presence of

$$(51) \quad \nabla_{12}^{33} = \nabla_{12}^{43} + \nabla_{12}^{34} = \nabla_{12}^{44} = 0 \ .$$

We note first of all that

$$\nabla_{12}^{33} = \nabla_{12}^{44} = 0$$

is equivalent to imposing the conditions

$$(52) \quad \begin{aligned} a^{13} &= \mu b^{13} \ , \quad a^{14} = \lambda b^{14} \ , \\ a^{23} &= \mu b^{23} \ , \quad a^{24} = \lambda b^{24} \ , \end{aligned}$$

where μ and λ are arbitrary factors, which are different from

zero, otherwise A_{12}^{34} would be zero. It follows that

$$\nabla_{12}^{43} + \nabla_{12}^{34} = (\mu - \lambda) B_{12}^{34}, \quad \Delta_{21}^{34} - \Delta_{12}^{34} = -(\mu + \lambda) B_{12}^{34}.$$

Since B_{12}^{34} is different from zero, the condition (51) is fulfilled if $\mu = \lambda$; we then have

$$\Delta_{22}^{34} = \Delta_{11}^{34} = 0, \quad \Delta_{12}^{34} + \Delta_{21}^{34} = 0, \quad \Delta_{12}^{34} - \Delta_{21}^{34} = 2\lambda B_{12}^{34}, \quad A_{12}^{34} = \lambda^2 B_{12}^{34}.$$

Consequently equation (48) becomes an identity, while the two families of developables of Γ_1 coincide and are determined by

$$dv + \lambda du = 0.$$

A conjugate net of ruled surfaces in Γ_1 is therefore not defined under these circumstances. A glance at system (C) in the presence of (52), with $\mu = \lambda$, shows that the point P_2 lies in the tangent plane of S_1 at the point P_1 , while P_1 lies in the tangent plane of S_2 at the point P_2 .

Let us next consider the relations (52), with $\mu = -\lambda$.

Then

$$\Delta_{22}^{34} = -2\lambda b^{23} b^{24}, \quad \Delta_{11}^{34} = -2\lambda b^{13} b^{14}, \quad A_{12}^{34} = -\lambda^2 B_{12}^{34},$$

and

$$\Delta_{12}^{34} = \Delta_{21}^{34} = -\lambda(b^{13} b^{24} + b^{14} b^{23}),$$

so that the developables of Γ_1 are determined by

$$dv^2 - \lambda^2 du^2 = 0,$$

and the focal points of ℓ_1 are given by

$$b^{23} b^{24} \omega^2 + (b^{13} b^{24} + b^{14} b^{23}) \omega + b^{13} b^{14} = 0,$$

whose discriminant is $(B_{12}^{34})^2 \neq 0$. Hence the conditions (52),

with $\mu = -\lambda$, imply

$$\nabla_{12}^{33} = \nabla_{12}^{44} = 0, \quad \Delta_{12}^{34} - \Delta_{21}^{34} = 0,$$

which are sufficient, but not necessary, conditions that the

parametric net of Γ_1 be conjugate. Evidently a necessary and

sufficient condition that the parametric ruled surfaces of Γ_1 form a conjugate net is given by

$$(53) \quad \Delta_{12}^{34} - \Delta_{21}^{34} = 0.$$

An application of the substitution

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

to the foregoing discussion gives similar results for the congruence Γ_2 . The developables of this congruence are determined by

$$A_{34}^{12} du^2 + (\Delta_{34}^{12} - \Delta_{43}^{12}) du dv + B_{34}^{12} dv^2 = 0,$$

while the focal points of ℓ_2 are given by $\chi^3 + \omega \chi^4$, where

$$\Delta_{44}^{12} \omega^2 + (\Delta_{34}^{12} + \Delta_{43}^{12}) \omega + \Delta_{33}^{12} = 0.$$

The points of tangency of any plane

$$\chi_1 - \kappa \chi_2 = 0$$

with the parametric ruled surfaces R_{2u} and R_{2v} of Γ_2 are respectively $\chi^3 + \omega_1 \chi^4$ and $\chi^3 + \omega_2 \chi^4$, where

$$\omega_1 = (\kappa a^{32} - a^{31}) / (a^{41} - \kappa a^{42}), \quad \omega_2 = (\kappa b^{32} - b^{31}) / (b^{41} - \kappa b^{42}).$$

Hence the parametric net is conjugate if, and only if,

$$a_2 \kappa^2 + \ell_2 \kappa + c_2 = 0,$$

where

$$a_2 = \nabla_{34}^{22} \{ \Delta_{43}^{12} - \Delta_{34}^{12} \} / \Delta_{44}^{12},$$

$$\ell_2 = - \{ (\nabla_{34}^{21})^2 - (\nabla_{34}^{12})^2 \} / \Delta_{44}^{12} = - \{ \nabla_{34}^{21} + \nabla_{34}^{12} \} \{ \Delta_{43}^{12} - \Delta_{34}^{12} \} / \Delta_{44}^{12},$$

$$c_2 = \nabla_{34}^{11} \{ \Delta_{43}^{12} - \Delta_{34}^{12} \} / \Delta_{44}^{12}.$$

Therefore a necessary and sufficient condition that the parametric ruled surfaces of Γ_2 form a conjugate net is

$$(54) \quad \Delta_{34}^{12} - \Delta_{43}^{12} = 0.$$

The conditions (53) and (54) are in general independent of each other; taken together, they provide a necessary and sufficient condition that the parametric ruled surfaces form a conjugate net in each of Γ_1 and Γ_2 .

Let Γ_1 and Γ_2 represent a pair of Green-reciprocal congruences, defined by system (F) and the equations

$$\rho = x_u - b x, \quad \sigma = x_v - a x.$$

We find

$$\begin{aligned} y_u &= (f + b m + a s) x + A y + m \rho + s \sigma, \\ y_v &= (g + b t + a n) x + B y + t \rho + n \sigma, \\ \rho_u &= F x + (\theta_u - b) \rho + \beta \sigma, \\ \rho_v &= h x + \gamma + a \rho, \\ \sigma_u &= k x + \gamma + b \sigma, \\ \sigma_v &= G x + \gamma \rho + (\theta_v - a) \sigma, \end{aligned}$$

where F, G, h, k are defined in §8. The correspondence between the generators of the reciprocal congruences Γ_1 and Γ_2 is defined by a system of equations of the form (C) in which we have written x, y, ρ, σ in place of x^1, x^2, x^3, x^4 respectively; the coefficients of this system are as follows:

$$\begin{aligned} a^{11} &= b, & a^{12} &= 0, & a^{13} &= 1, & a^{14} &= 0, \\ a^{21} &= f + b m + a s, & a^{22} &= A, & a^{23} &= m, & a^{24} &= s, \\ a^{31} &= F, & a^{32} &= 0, & a^{33} &= \theta_u - b, & a^{34} &= \beta, \\ (55) \quad a^{41} &= h, & a^{42} &= 1, & a^{43} &= 0, & a^{44} &= b, \end{aligned}$$

$$\begin{aligned} b'' &= a, & b^{12} &= 0, & b^{13} &= 0, & b^{14} &= 1, \\ b^{21} &= q + b^1 t + a m, & b^{22} &= B, & b^{23} &= t, & b^{24} &= m, \\ b^{31} &= h, & b^{32} &= 1, & b^{33} &= a, & b^{34} &= 0, \\ b^{41} &= \Theta, & b^{42} &= 0, & b^{43} &= \gamma, & b^{44} &= \Theta_N - a. \end{aligned}$$

Therefore

$$\Delta_{12}^{34} - \Delta_{21}^{34} = m - m = a_u - b_N,$$

$$\Delta_{34}^{12} - \Delta_{43}^{12} = h - h = a_u - b_N.$$

The conditions (53) and (54) thus agree with (44).

An intersector surface of a congruence Γ_1 , with respect to a congruence Γ_2 , where Γ_1 and Γ_2 are characterized by a system of differential equations of the form (C), has been defined by Cook as a transversal surface of Γ_1 such that the tangent plane to the surface at the point of intersection with ℓ_1 contains the corresponding generator ℓ_2 of Γ_2 . If a one-parameter family of such surfaces exists for the congruence Γ_1 , then Γ_1 is said* to possess property D with respect to Γ_2 . Any intersector surface of Γ_1 with respect to Γ_2 is defined by

$$z = \chi^1 + \lambda \chi^2,$$

where $\lambda = \lambda(u, v)$ is an arbitrary solution of the Riccati system of differential equations

$$\begin{aligned} \lambda_u &= a^{21} \lambda^2 + (a^{11} - a^{22}) \lambda - a^{12}, \\ \lambda_v &= b^{21} \lambda^2 + (b^{11} - b^{22}) \lambda - b^{12}. \end{aligned} \quad (56)$$

*Cook, loc. cit., p. 34.

It has been shown* that a necessary and sufficient condition for the existence of a one-parameter set of solutions λ of (56) is given by

$$(57) \quad \begin{aligned} a^{13}b^{32} + a^{14}b^{42} &= b^{13}a^{32} + b^{14}a^{42}, \\ a^{23}b^{31} + a^{24}b^{41} &= b^{23}a^{31} + b^{24}a^{41}, \\ a^{13}b^{31} + a^{14}b^{41} - a^{23}b^{32} - a^{24}b^{42} &= b^{13}a^{31} + b^{14}a^{41} - b^{23}a^{32} - b^{24}a^{42}. \end{aligned}$$

Likewise Γ_2 possesses property $\mathfrak{J}$ with respect to Γ_1 if, and only if,

$$(58) \quad \begin{aligned} a^{31}b^{14} + a^{32}b^{24} &= b^{31}a^{14} + b^{32}a^{24}, \\ a^{41}b^{13} + a^{42}b^{23} &= b^{41}a^{13} + b^{42}a^{23}, \\ a^{31}b^{13} + a^{32}b^{23} - a^{41}b^{14} - a^{42}b^{24} &= b^{31}a^{13} + b^{32}a^{23} - b^{41}a^{14} - b^{42}a^{24}. \end{aligned}$$

The two sets of conditions (57) and (58) provide necessary and sufficient conditions that each congruence possess property $\mathfrak{J}$ with respect to the other.

We shall now apply this concept of property $\mathfrak{J}$ to an arbitrary pair of reciprocal congruences Γ_1, Γ_2 . It has already been noted that the correspondence between the generators of a pair of reciprocal congruences is defined by a system of differential equations of the form (C), whose coefficients are given by (55). We find that the first of conditions (57) is satisfied identically by the coefficients (55), while the second and third of conditions (57) reduce respectively to

$$\begin{aligned} m\hbar + s\mathfrak{G} &= tF + n\hbar, \\ a_u &= b_v, \end{aligned}$$

which are equivalent to

$$(59) \quad \beta F(\psi_2 - 2b) = \beta \mathfrak{G}(\psi_2 - 2a), \quad a_u = b_v.$$

*Cook, loc. cit., p. 35.

Hence a sufficient condition that Γ_1 possess property $\mathfrak{D}$ with respect to Γ_2 is given by

$$(60) \quad a = \psi_2/2, \quad b = \psi_1/2, \quad a_u = b_v.$$

The conditions (60) imply

$$(61) \quad (\log \beta/\gamma)_{uv} = 0,$$

in which case the reference surface S_x is said to be isothermally-asymptotic.* We recall that the parametric net of ruled surfaces of Γ_1 intersects S_x in the asymptotic curves thereon; for convenience in what follows, we shall denote this particular net of ruled surfaces in Γ_1 by $\bar{\mathcal{N}}_1$, while the corresponding net in Γ_2 will be denoted by $\bar{\mathcal{N}}_2$. It follows from (60) that Γ_1 possesses property $\mathfrak{D}$ with respect to Γ_2 if Γ_1 and Γ_2 are directrix congruences of Wilczynski in which $\bar{\mathcal{N}}_1$ and $\bar{\mathcal{N}}_2$ are conjugate nets.

If a general congruence Γ_1 possesses property $\mathfrak{D}$ with respect to a second congruence Γ_2 , and if the developables of Γ_1 trace out a conjugate net of curves on every intersector surface, then Γ_1 is said[†] to possess property $\mathfrak{D}_1$ with respect to Γ_2 . Fubini has shown[‡] that when Γ_1 possesses property $\mathfrak{D}$ with respect to Γ_2 , the developables of Γ_1 cannot trace out a conjugate net on any of the intersector surfaces unless Γ_1 possesses property $\mathfrak{D}_1$ with respect to Γ_2 . Returning now to

*Fubini and Čech, Geometria Proiettiva Differenziale, Zanichelli, Bologna, Vol. I, p. 94.

†Cook, loc. cit., p. 40.

‡G. Fubini, Su alcune classi di congruenze di rette e sulle trasformazioni delle superficie R, Annali di Matematica, Serie IV, Vol. I (1923-24), p. 244. See also Cook, loc. cit., p. 40.

the pair of reciprocal congruences, we observe that S_x is an intersector surface of Γ_1 with respect to Γ_2 ; further, the conditions (59) are necessary and sufficient conditions that Γ_1 possess property $\mathfrak{D}$ with respect to Γ_2 , with the additional property that the developables of Γ_1 intersect S_x in the curves of a conjugate net. Hence Γ_1 possesses property $\mathfrak{D}_1$ with respect to Γ_2 if, and only if, the relations (59) hold.

Theorem: Let Γ_1 and Γ_2 be directrix congruences of Wilczynski. Then Γ_1 possesses property $\mathfrak{D}_1$ with respect to Γ_2 if, and only if, the nets $\tilde{\mathfrak{N}}_1, \tilde{\mathfrak{N}}_2$ are conjugate. As a corollary we might add that every intersector surface of Γ_1 with respect to Γ_2 possesses property $\mathfrak{L}$ with respect to the net $\tilde{\mathfrak{N}}_1$.

We now proceed to investigate the conditions under which Γ_2 may possess property $\mathfrak{D}$ with respect to Γ_1 , when Γ_1 and Γ_2 are Green-reciprocal congruences. In the presence of (55), conditions (58) reduce to

$$(62) \quad a = \psi_2/2, \quad b = \psi_1/2, \quad 2\beta\gamma = (\log \beta\gamma)_{uv},$$

which are therefore necessary and sufficient conditions that

Γ_2 possess property $\mathfrak{D}$ with respect to Γ_1 . By combining (59) and (62), we find that each of Γ_1, Γ_2 possesses property $\mathfrak{D}$ with respect to the other in case

$$(63) \quad a = \psi_2/2, \quad b = \psi_1/2, \quad \beta\gamma = (\log \beta)_{uv} = (\log \gamma)_{uv}.$$

If the curve $v = \text{const.}$ on S_x belongs to a linear complex, then

$$\beta\gamma = (\log \beta)_{uv}.$$

Similarly, a necessary condition that the curve $u = \text{const.}$ on

S_x belong to a linear complex is given* by

$$(\beta\gamma) = (\log \gamma)_{uv}.$$

We conclude from (63) that the directrix congruences of Wilczynski are the only reciprocal congruences of which it is possible to say that either one possesses property $\mathfrak{D}$ with respect to the other; a pair of directrix congruences are related in this manner if each of the two asymptotic curves through any point of the reference surface S_x belongs to a linear complex, in which case S_x is isothermally-asymptotic.

*Fubini and Čech, loc. cit., Vol. I, p. 115.

10. THE ASSOCIATED CONGRUENCE

The tangent planes at corresponding points of the reference surfaces, S_x and S_y , of a congruence Γ_{xy} defined by system (F), intersect in a line ℓ which joins the points r_1 and r_2 , where

$$(64) \quad \begin{aligned} r_1 &= y_u - Ay = f\alpha + m\alpha_u + s\alpha_v, \\ r_2 &= y_v - By = g\alpha + t\alpha_u + n\alpha_v. \end{aligned}$$

As the line ℓ_{xy} generates the congruence Γ_{xy} , the line ℓ generates a congruence Γ ; to any net $\mathcal{N}_1$ of ruled surfaces in Γ_{xy} there corresponds a net $\mathcal{N}$ in Γ . As in the preceding section, $\bar{\mathcal{N}}_1$ will denote the net of ruled surfaces in Γ_{xy} which intersects S_x in the asymptotic curves on the latter, while $\bar{\mathcal{N}}$ will represent the corresponding net in Γ . Since ℓ_{xy} and ℓ are skew lines in one-to-one correspondence, then Γ_{xy} and Γ constitute a pair of congruences of the type defined by system (C). It has been pointed out by Lane that there exists a unique quadric surface Q_x which has contact of the second order with S_x at P_x and contact of the first order with S_y at P_y . It follows that the lines ℓ_{xy} and ℓ are reciprocal polars with respect to Q_x . Thus Γ_{xy} and Γ may be regarded as reciprocal congruences, but not in the sense of Green, since Q_x replaces the osculating quadric of S_x at P_x . The quadric Q_x and a similarly defined quadric Q_y have been called* the associated quadrics of Γ_{xy} . For this reason we have ventured to call Γ

*E. P. Lane, On the Fundamental Transformation of Surfaces, Annals of Mathematics, Second Series, Vol. 30 (1929), p. 460.

the associated congruence of Γ_{xy} .

In the preceding section it was seen that the directrix congruences of Wilczynski play an important part in the linking up of the concept of property $\mathfrak{J}$ with the notion of conjugate nets of ruled surfaces. It is not surprising, in view of the polar relationship between ℓ_{xy} and ℓ , that a similar rôle is played by a pair of congruences Γ_{xy} and Γ whose developables correspond to the same net of curves on a reference surface. We shall first show that the conjugacy of the parametric nets, $\bar{\mathfrak{N}}_1$ and $\bar{\mathfrak{N}}$, provides a sufficient condition that the developables of Γ_{xy} and Γ correspond to the same net of curves on S_x or S_y . This theorem is not valid for a pair of Green-reciprocal congruences. The central theorem of this section is that Γ_{xy} possesses property $\mathfrak{J}_1$ with respect to Γ , if and only if the parametric ruled surfaces form a conjugate net in each of Γ_{xy} and Γ . In the concluding paragraph of the section, a rather interesting point is located on each line ℓ of the congruence Γ .

To obtain the coefficients of the particular system (C) which characterizes the pair of congruences Γ_{xy} and Γ , we first solve equations (64) for x_u , x_v , y_u , y_v , each in terms of x , y , r_1 , and r_2 :

$$(65) \quad \begin{aligned} x_u &= \bar{A}x + m r_1 / \Delta - s r_2 / \Delta, & y_u &= A y + r_1, \\ x_v &= \bar{B}x - t r_1 / \Delta + m r_2 / \Delta, & y_v &= B y + r_2, \end{aligned}$$

where the dashes indicate coefficients of the system $(\bar{G})$ which is symmetric with system (F) in x and y ; these coefficients are obtained from equations (1) and (2) by replacing g_{ij} by the

functions f_{ij} defined in §4. By differentiation and making use of the equations of system (F) we also obtain from (64) the following equations:

$$\begin{aligned}
 r_{1u} &= f_{13}x + f_{11}x_u + f_{12}x_v + s y, \\
 r_{2u} &= f_{32}x + f_{23}x_u + f_{34}x_v + m y, \\
 r_{1v} &= f_{23}x + f_{21}x_u + f_{22}x_v + m y, \\
 r_{2v} &= f_{42}x + f_{43}x_u + f_{44}x_v + t y.
 \end{aligned}
 \tag{66}$$

If we substitute in (66) the values of x_u and x_v from (65), we obtain equations which express r_{1u} , r_{2u} , r_{1v} , r_{2v} , each in terms of x , y , r_1 , and r_2 ; the latter equations, along with (65), constitute the required system (C). The coefficients of this system are as follows:

$$\begin{aligned}
 a^{11} &= \bar{A}, \quad a^{12} = 0, \quad a^{13} = m/\Delta, \quad a^{14} = -s/\Delta, \\
 a^{21} &= 0, \quad a^{22} = A, \quad a^{23} = 1, \quad a^{24} = 0, \\
 a^{31} &= \bar{L}, \quad a^{32} = s, \quad a^{33} = \bar{a} - A, \quad a^{34} = \bar{\beta}, \\
 a^{41} &= \bar{M}, \quad a^{42} = m, \quad a^{43} = \bar{a} - B, \quad a^{44} = \bar{b}, \\
 b^{11} &= \bar{B}, \quad b^{12} = 0, \quad b^{13} = -t/\Delta, \quad b^{14} = m/\Delta, \\
 b^{21} &= 0, \quad b^{22} = B, \quad b^{23} = 0, \quad b^{24} = 1, \\
 b^{31} &= \bar{M}, \quad b^{32} = m, \quad b^{33} = \bar{a}, \quad b^{34} = \bar{b} - A, \\
 b^{41} &= \bar{N}, \quad b^{42} = t, \quad b^{43} = \bar{\gamma}, \quad b^{44} = \bar{s} - B.
 \end{aligned}
 \tag{67}$$

The developables of Γ are defined by

$$(68) \quad (m\bar{L} - s\bar{M})du^2 + \{t\bar{L} - s\bar{N} + \bar{M}(n-m)\}dudv - (m\bar{N} - t\bar{M})dv^2 = 0,$$

in which we shall assume that the coefficients of du^2 and dv^2 are different from zero, so that the parametric surfaces of Γ will be non-developable. As a consequence of (67), we find

$$\Delta_{12}^{34} - \Delta_{21}^{34} = (n-m)/\Delta,$$

(69)

$$\Delta_{34}^{12} - \Delta_{43}^{12} = t\bar{L} - s\bar{N} + \bar{M}(n-m).$$

Therefore the nets $\bar{\mathcal{N}}_1$ and $\bar{\mathcal{N}}$ are both conjugate in case

(70)

$$n=m, \quad t\bar{L} - s\bar{N} = 0.$$

The asymptotics on S_y are determined by the equation

(71)

$$\bar{L} du^2 + 2\bar{M} du dv + \bar{N} dv^2 = 0,$$

whose discriminant is assumed to be different from zero, so

that S_y is non-developable. It is to be observed that the

harmonic invariant of (68) and (71) is given by

$$(n-m)(\bar{L}\bar{N} - \bar{M}^2),$$

while the harmonic invariant of (23) and (71) is given by

$$\Delta_{43}^{12} - \Delta_{34}^{12}.$$

Hence we may state the following double theorem: the develop-

ables of Γ correspond to a conjugate net of curves on S_y (S_x)

if, and only if, the developables of Γ_{xy} intersect S_x (S_y) in

a conjugate net of curves. In the presence of (70), equations

(23) and (68) both reduce to

(72)

$$s du^2 - t dv^2 = 0.$$

We conclude that if the parametric ruled surfaces form a con-

jugate net in each of Γ_{xy} and Γ , then the developables of Γ_{xy}

and Γ correspond to the same net of curves on S_x or S_y ; the

curves of this net form a conjugate net on both reference sur-
faces.

The focal points of a generator ℓ are given by $r_1 + \omega r_2$, where

$$(73) \quad (t\bar{M} - m\bar{N})\omega^2 + \{t\bar{L} - s\bar{N} - \bar{M}(n-m)\}\omega + (m\bar{L} - s\bar{M}) = 0.$$

If (70) holds, then (73) reduces to

$$t\omega^2 - s = 0.$$

With the hypothesis that the parametric nets $\bar{\mathcal{N}}_1$ and $\bar{\mathcal{N}}$ are both conjugate, we have the following conclusions:

- i) the surfaces S_{r_1} and S_{r_2} are point-conjugates with respect to Γ .
- ii) the developables of Γ_{xy} and Γ correspond to the same net of curves (72); the tangents at any point of S_x (or S_y) to the curves of this net intersect the corresponding line ℓ in the focal points of that line.

A necessary and sufficient condition that Γ_{xy} possess property $\mathcal{J}$ with respect to Γ is that the coefficients (67) satisfy the relations (57); these relations are also necessary and sufficient conditions for the existence of a one-parameter set of functions

$$\lambda = C e^{\int (\bar{A} - A) du + \int (\bar{B} - B) dv},$$

where C is a parameter, such that any intersector surface of Γ_{xy} with respect to Γ is given by $z = x + \lambda y$. The first two of the relations (57) are satisfied identically by the coefficients (67), while the third relation reduces to

$$(74) \quad t\bar{L} - s\bar{N} + \bar{M}(n-m) + \Delta(n-m) = 0.$$

Therefore (70) yields a sufficient condition that Γ_{xy} possess property $\mathcal{J}$ with respect to Γ . Further, the conditions (70)

are both necessary and sufficient that Γ_{xy} possess property $\mathcal{J}$, with respect to Γ , with the additional property that the developables of Γ_{xy} intersect S_x in the curves of a conjugate net. Since S_x is an intersector surface of Γ_{xy} with respect to Γ , the developables of Γ_{xy} must in that event trace out a conjugate net of curves on every intersector surface of Γ_{xy} .

Theorem: The congruence Γ_{xy} possesses property $\mathcal{J}_1$ with respect to the associated congruence Γ , if and only if the parametric ruled surfaces form a conjugate net in each of Γ_{xy} and Γ .

We have previously seen that if the parametric net of ruled surfaces in each of Γ_{xy} and Γ is conjugate, then the developables of the congruences correspond. A theorem of Fubini's follows as a consequence: if Γ_{xy} possesses property $\mathcal{J}_1$ with respect to Γ , then the developables of the congruences correspond.*

The following conditions are necessary and sufficient that Γ possess property $\mathcal{J}$ with respect to Γ_{xy} :

$$\begin{aligned} m\bar{L} + \rho\bar{M} &= -\rho\Delta, \\ (75) \quad t\bar{M} + n\bar{N} &= -t\Delta, \\ t\bar{L} + (m+n)\bar{M} + \rho\bar{N} &= -(m+n)\Delta. \end{aligned}$$

The determinant of the coefficients of $\bar{L}$, $\bar{M}$, $\bar{N}$ in (75) is $-(m+n)\Delta$, which is different from zero provided S_x and S_y are not point-conjugates with respect to Γ_{xy} . With this exceptional case noted, we may solve (75) and obtain

*Fubini, loc. cit., p. 246. Also Cook, loc. cit., p. 40.

$$(76) \quad \bar{L} = \bar{N} = 0, \quad \bar{M} = -\Delta.$$

The values (76) satisfy equation (74). Moreover, in the presence of (76), we have

$$\Delta_{34}^{12} - \Delta_{43}^{12} = \Delta^2 (\Delta_{21}^{34} - \Delta_{12}^{34}).$$

Hence if Γ possesses property $\mathcal{J}$ with respect to Γ_{xy} and if the reference surfaces S_x, S_y are not point-conjugates with respect to Γ_{xy} , then i) Γ_{xy} possesses property $\mathcal{J}$ with respect to Γ ; ii) the asymptotics of S_x and S_y correspond; iii) the conjugacy of $\bar{\mathcal{H}}$ is equivalent to the conjugacy of $\bar{\mathcal{H}}_1$.

There exists on each line ℓ a point which is closely related to the corresponding associated quadric Q_x of Γ_{xy} . The equation of ℓ is

$$x_4 = x_1 + \bar{A}x_2 + \bar{B}x_3 = 0,$$

while the equation of the line $\ell_{\rho\sigma}$, which is the polar line of ℓ_{xy} with respect to the quadric (41), is given by

$$x_4 = x_1 + bx_2 + ax_3 = 0.$$

The lines ℓ and $\ell_{\rho\sigma}$ coincide if the coefficients of system (F) satisfy the relations

$$(77) \quad \begin{aligned} f + bm + as &= 0, \\ g + bt + an &= 0. \end{aligned}$$

If ℓ and $\ell_{\rho\sigma}$ do not coincide they intersect in a point R whose local coordinates are

$$\begin{aligned} x_1 &= a\bar{A} - b\bar{B}, \\ x_2 &= \bar{B} - a, \\ x_3 &= b - \bar{A}, \\ x_4 &= 0. \end{aligned}$$

The associated quadric Q_x is given by

$$x_1 x_4 - x_2 x_3 + \bar{A} x_2 x_4 + \bar{B} x_3 x_4 = 0.$$

The quadric of Lie of S_x at P_x , whose equation is given by (41), has in common with Q_x the asymptotics of S_x at P_x . The residual intersection is therefore a conic, which lies in the plane

$$(78) \quad (b - \bar{A})x_1 + (a - \bar{B})x_3 - \{ab + c - (\beta\gamma + \theta_{uv})/2\}x_4 = 0.$$

The line of intersection of the plane (78) with the tangent plane of S_x at P_x passes through the point R.

11. COVARIANT TRANSVERSAL SURFACES

Any non-ruled surface S determines a covariant congruence, namely, the congruence of projective normals of S . We have seen that in this congruence the ruled surfaces which trace out the asymptotics on S form a conjugate net. It is the aim of the present section to show that if a congruence is given, whose focal surfaces are known, then it is possible to construct a covariant transversal surface of the congruence. It will be demonstrated in the next section of this paper that the construction of such a surface makes possible the determination, by purely geometric means, of a conjugate net of ruled surfaces in the congruence.

Two methods of obtaining covariant transversal surfaces are suggested by the fact that the developables of a congruence circumscribe the focal surfaces of the congruence in the curves of a conjugate net. We shall employ these methods in connection with a congruence which will be thought of as the conjugate congruence of two surfaces, S_x and S_y , in the relation of a fundamental transformation. It is important to observe that this type of congruence does not place any restriction on the generality of the two methods; it does, however, yield a simplification of the analytic representation of the covariant transversal surfaces.

A congruence Γ_{xy} , whose reference surfaces S_x and S_y are in the relation of a fundamental transformation with respect to the congruence, is defined by system (L). We shall

first present a system of differential equations which defines, except for a projective transformation, the focal surface S_f of Γ_{xy} , where

$$\xi = m\alpha - y.$$

By differentiation and making use of the equations of system (L), we find

$$\xi_u = m_u \alpha + (m-m')\alpha_u = (m-m')(\alpha_u - b\alpha) = (m-m')\alpha_{-1},$$

$$\xi_v = m_v \alpha,$$

$$\xi_{uv} = m_{uv} \alpha + m_v \alpha_u,$$

$$\xi_{uu} = \{m_{uu} + \mu(m-m')\}\alpha + \{2m_u - m_u + d(m-m')\}\alpha_u + \beta(m-m')\alpha_v + L(m-m')y,$$

$$\xi_{vv} = m_{vv} \alpha + m_v \alpha_v.$$

The focal surface S_f is therefore projectively characterized by the differential equations

$$(79) \quad \begin{aligned} \xi_{uv} &= b' \xi_u + c' \xi_v + d' \xi, \\ \xi_{uu} &= a'' \xi_{vv} + b'' \xi_u + c'' \xi_v + d'' \xi, \end{aligned}$$

where

$$b' = m_v / (m-m'), \quad c' = h(m-m') / m_v, \quad d' = 0,$$

$$a'' = \beta(m-m') / m_v, \quad b'' = \alpha - 2b - m_u / (m-m'),$$

$$c'' = \beta(m-m') \{ \alpha - \beta_v / \beta - \delta - m_{vv} / m_v \} / m_v,$$

$$d'' = L(m-m'), \quad h = ab - b_v.$$

The system of differential equations (79) was used by Green as a basis for a study of the projective theory of conjugate nets of curves on a surface.* The integrability conditions of

*G. M. Green, Projective Differential Geometry of One-Parameter Families of Space Curves, and Conjugate Nets on a Curved Surface, American Journal of Mathematics, Vol. 37 (1915).

(79) show the existence of a function $\phi(u, v)$ such that

$$4\phi_u = b'' + 2c' = \alpha + 2m_{uv}/m_v + m_u/(m-m),$$

$$4\phi_v = (2a''b' - c'' - a''_{vv})/a'' = \delta + 2m_{vv}/m_v + m_v/(m-m).$$

A canonical form for system (79) has been obtained by Green, through a transformation of the form

$$(80) \quad \xi = \lambda \bar{\xi},$$

where

$$(81) \quad \lambda = e^{\phi}.$$

If the coefficients of this canonical form are denoted by capital letters then the form is characterized by the conditions

$$B'' + 2C' = 0, \quad 2A''B' - C'' - A''_{vv} = 0,$$

where

$$B' = b' - \phi_v = 3m_v/4(m-m) - \delta/4 - m_{vv}/2m_v,$$

$$C' = c' - \phi_u = m_u/4(m-m) - \alpha/4 + m_{uv}/2m_v + b,$$

$$D' = d' + b'\phi_u + c'\phi_v - \phi_{uv} - \phi_u\phi_v,$$

$$A'' = a'' = \beta(m-m)/m_v,$$

$$B'' = b'' - 2\phi_u = \alpha/2 - 2b - m_{uv}/m_v - m_u/2(m-m),$$

$$C'' = c'' + 2a''\phi_v = \beta(m-m)\{a - \beta_v/\beta - \delta/2 + m_v/2(m-m)\}/m_v,$$

$$D'' = d'' + b''\phi_u + c''\phi_v - \phi_{uu} + a''\phi_{vv} - \phi_u^2 + a''\phi_v^2.$$

Lane has shown* that the transformation (80), with λ determined by (81), leads to a new form of system (79), which

*E. P. Lane, Contributions to the Theory of Conjugate Nets, American Journal of Mathematics, Vol. 49 (1927).

is written with covariant derivatives as follows:

$$f_{12} = B' f_1 + C' f_2 + D' f,$$

$$f_{11} = A'' f_{22} - 2C' f_1 + 2A'' B' f_2 + D'' f,$$

where B' and C' are fundamental invariants of (79) and are given by

$$B' = B' - 3A''_{\nu\nu}/8A'' = 3m_{\nu\nu}/8(m-m) - 3\beta_{\nu\nu}/8\beta - 5/4 + 3a/8 - m_{\nu\nu\nu}/8m_{\nu\nu},$$

$$C' = C' + A''_{uu}/8A'' = m_{uu}/8(m-m) + \beta_{uu}/8\beta - a/4 + 7b/8 + 3m_{u\nu\nu}/8m_{\nu\nu}.$$

It is further shown that the choice of proportionality factor (81) yields Fubini's normal coordinates, and, by means of the second and first differential parameters, the projective normal through any point of the focal surface and the reciprocal of the projective normal, with respect to the quadric of Lie, are obtained.

We are now in a position to describe the first of the two methods of constructing a covariant transversal surface of a congruence whose focal surfaces are known. When the co-ordinates are Fubini's normal coordinates, the reciprocal of the projective normal through the point f of the surface defined by (79) is the line joining the points f_u and $f_{\nu\nu}$.*

*Lane, loc. cit., p. 569.

Hence for the coordinate system employed in system (L) the reciprocal of the projective normal through ξ is the line joining the points

$$\xi_u - \phi_u \xi, \quad \xi_v - \phi_v \xi.$$

The generators ℓ_{xy} of Γ_{xy} and the reciprocals of the projective normals of the focal surface S_f are in one-to-one correspondence; moreover, corresponding lines are coplanar. If we define

$$\Phi \equiv m_v - n \phi_v,$$

then the reciprocal of the projective normal through ξ and the corresponding generator ℓ_{xy} intersect in the point

$$(82) \quad P_1 = \Phi x + \phi_v y = \{m_v + (m-n)\phi_v\}x - \phi_v \eta.$$

The totality of points P_1 represents a covariant transversal surface S_1 of Γ_{xy} . A second surface of this type may of course be obtained by repeating the foregoing construction on the focal surface S_η of Γ_{xy} . The reference surface S_x coincides with S_1 in case the coefficients of system (L) satisfy the relation

$$\delta = m_v / (m-n) - 2m_{vv} / m_v.$$

Similarly, S_y coincides with S_1 in case

$$m = U e^\phi,$$

where U is an arbitrary function of u .

The second method of obtaining a covariant transversal surface of Γ_{xy} differs from the first method in that the reciprocal of the projective normal through ξ is replaced by the ray of ξ with respect to the net of the projective lines of curvature of S_f . The generators of Γ_{xy} are in one-to-one

correspondence with the generators of the ray congruence of the net of the projective lines of curvature of S_f , and corresponding generators are coplanar. Hence the totality of points of intersection of generators of Γ_{xy} with corresponding rays represents a transversal surface which is defined geometrically in terms of the congruence Γ_{xy} . We shall denote by S_2 a transversal surface which is determined in this manner, and shall proceed to the analytic determination of any point P_2 of S_2 .

The net of the projective lines of curvature of S_f is defined* by the differential equation

$$(83) \quad \{\beta L'(n-m)/m_{\nu}\} du^2 - M' du dv - L' dv^2 = 0,$$

where

$$L' = D' + \alpha'_{\nu} + B'_u + 2C'B' + 2B'\alpha' - 4B'\alpha',$$

$$M' = D'' + A''(+B'B' - 4B'^2 + 2B'_{\nu}) + A''_{\nu}B' - 4C'\alpha' + 4\alpha'^2 - 2\alpha'_u + \alpha'A''_u / A''.$$

We shall assume that the discriminant of (83) does not vanish, so that the focal surfaces of the projective normal congruence of S_f do not coincide. At each point of S_f the equation (83) determines the tangents to the two projective lines of curvature which pass through that point. Let us denote by ζ_1 a point of the first, and by ζ_2 a point of the second of these conjugate tangents; then

$$\zeta_i = \mu_i \xi + \xi_u + \lambda_i \xi_{\nu} \quad (i=1,2),$$

where $\mu_i(u, \nu)$ is a parameter and $\lambda_i = (dv/du)_i$ is a root of (83). As the point ξ varies along a curve of the second

*Lane, loc. cit., p. 569.

family of the net (83), that is, a curve whose direction at any point is given by λ_2 , the point ξ_1 will describe a curve, any point on the tangent of which is given by

$$\begin{aligned}\xi_1' &= \xi_{1u} + \lambda_2 \xi_{1v} \\ &= (d'' + \mu_{1u} + \lambda_2 \mu_{1v})\xi + \{\mu_1 + b'' + b'(\lambda_1 + \lambda_2)\}\xi_u \\ &\quad + \{c'' + \lambda_{1u} + \lambda_2 \lambda_{1v} + \mu_1 \lambda_2 + c'(\lambda_1 + \lambda_2)\}\xi_v.\end{aligned}$$

Hence ξ_1 will be a ray point of a curve of the second family of the net (83), corresponding to the point ξ , if there exists a function μ_1 such that the line $\xi\xi_1$ coincides with the line $\xi\xi_1'$. We find for the desired value of μ_1 :

$$(84) \quad \mu_1 = \{(\lambda_1 + \lambda_2)(c' - \lambda_1 b') + c'' - \lambda_1 b'' + \lambda_{1u} + \lambda_2 \lambda_{1v}\} / (\lambda_1 - \lambda_2).$$

By interchanging λ_1 and λ_2 in (84) we obtain an expression for μ_2 such that ξ_2 will be a ray point of a curve of the first family of the net (83), corresponding to the point ξ .

Referred to a local tetrahedron of reference, with vertices at the points $\xi, \xi_u, \xi_v, \xi_{uv}$, and a suitably chosen unit point, we find that the local coordinates of any point on the ray of ξ with respect to the net (83) are given by

$$\begin{aligned}\xi_1 &= \{(\lambda_1 + \lambda_2)(c' - \lambda_1 b') + c'' - \lambda_1 b'' + \lambda_{1u} + \lambda_2 \lambda_{1v}\} \\ &\quad + \mu \{(\lambda_1 + \lambda_2)(c' - \lambda_2 b') + c'' - \lambda_2 b'' + \lambda_{2u} + \lambda_1 \lambda_{2v}\}, \\ (85) \quad \xi_2 &= \lambda_1 - \lambda_2 + \mu(\lambda_2 - \lambda_1), \\ \xi_3 &= \lambda_1(\lambda_1 - \lambda_2) + \mu \lambda_2(\lambda_2 - \lambda_1), \\ \xi_4 &= 0,\end{aligned}$$

where μ is a parameter. Since $\xi_v = m_v x$, the edge $\xi\xi_v$ of the tetrahedron of reference coincides with the generator l_{xy}

of Γ_{xy} . Therefore the ray of ξ with respect to the net (83) intersects ℓ_{xy} in a point P_2 whose coordinates are given by (85) when $\mu=1$.

The equations which define the transformation of co-ordinates from the tetrahedron $\xi, \xi_u, \xi_v, \xi_{vv}$ to the tetrahedron x, x_u, x_v, y are as follows:

$$\begin{aligned} x_1 &= m \xi_1 + m_u \xi_u + m_v \xi_v + m_{vv} \xi_{vv}, \\ x_2 &= (m-m') \xi_2, \\ x_3 &= m_v \xi_v, \\ x_4 &= -\xi_1, \end{aligned} \quad (86)$$

the determinant of the transformation being $m_v^2(m-m')$, which is different from zero. Hence the coordinates of the point P_2 , referred to the tetrahedron x, x_u, x_v, y , are obtained by applying the transformation (86) to the coordinates (85) in which $\mu=1$. For convenience, let us define

$\Psi \equiv \{L'M'(\ell''-2c') - \ell' M'^2 - L'M'_u + M'L'_u - L'^2(a''_v - 2c'')\} / m_v(4a''L'^2 + M'^2)$, in which $4a''L'^2 + M'^2$ is the discriminant of (83). Then the coordinates of P_2 , referred to the tetrahedron x, x_u, x_v, y , are

$$(87) \quad x_1 = m\Psi + 1, \quad x_2 = x_3 = 0, \quad x_4 = -\Psi.$$

Hence

$$P_2 = \Psi\xi + \alpha.$$

If the surface S_ξ is referred to its projective lines of curvature then $L'=0$ and $M' \neq 0$. In this case, $\Psi = 1/(m-m')$, and the ray of the point ξ with respect to the net of the

projective lines of curvature of S_{ξ} is the line joining the points

$$\rho = \xi_u - \kappa' \xi = (m - \bar{m})(n_{\mathcal{N}} x_{-1} - h \xi) / n_{\mathcal{N}},$$

and

$$\sigma = \xi_{\mathcal{N}} - h' \xi = n_{\mathcal{N}}(m x - y) / (m - n);$$

hence P_2 coincides with the focal point $\eta = m x - y$.

12. A GEOMETRIC CONSTRUCTION FOR A CONJUGATE NET OF RULED SURFACES IN A CONGRUENCE

The purpose of this section is to show that the lines of a congruence can be grouped by geometric means into two one-parameter families of ruled surfaces which form a conjugate net. The method is applicable to a congruence whose developable surfaces are known and whose focal surfaces are non-developable.

Let us consider the congruence of lines which are tangent to two distinct, non-developable surfaces S_1 and S_2 . We have seen in §11 that it is possible to construct at least two covariant transversal surfaces for the congruence under consideration; let us assume that one of these surfaces, say S_1 , has been so constructed. The developables of the congruence intersect S_1 in a net of curves. If S_1 is non-developable it possesses a net of asymptotics. An asymptotic curve through any point P_1 of S_1 cannot coincide with either of the curves through P_1 which are traced out on S_1 by the developables of the congruence. The asymptotic tangents at P_1 and the tangents at P_1 to the two curves which the developables trace out through P_1 on S_1 determine an involution. The double rays of this involution can be determined by any one of a number of geometric constructions in the tangent plane of S_1 at P_1 .^{*} In general there is thus established at each point of S_1 a pair of lines which separate harmonically the asymptotic tangents

^{*}L. Cremona, Elements of Projective Geometry, Oxford (1913), pp. 102, 166.

at the point and also separate harmonically the tangents at the point to the curves which the developables trace out on S_1 . This two-parameter family of line-pairs therefore envelopes a conjugate net of curves on S_1 , and to each curve of this net there corresponds a ruled surface in the congruence. We thus obtain a net $\mathcal{N}$ of ruled surfaces in the congruence. The surface S_1 possesses property $\mathcal{L}$ with respect to the net $\mathcal{N}$; therefore $\mathcal{N}$ is conjugate. We proceed to the analytic determination of this net.

A congruence Γ_{xy} defined by system (L) satisfies the requirements of this section. We have seen that the covariant transversal surface S_1 of Γ_{xy} is defined by (82), namely

$$z = \Phi x + \phi_v y,$$

in which we have replaced P_1 by z . By means of the equations of system (L) and the integrability conditions of that system, it is possible to express uniquely every derivative of z as a linear combination of x , x_u , x_v and y . Let us write

$$z_{ij} \equiv \frac{\partial^2 z}{\partial u^i \partial v^j} = \alpha_{ij} x + \beta_{ij} x_u + \gamma_{ij} x_v + \delta_{ij} y \quad (i, j = 0, 1, 2, \dots).$$

We find

$$\alpha_{10} = \Phi_u, \quad \beta_{10} = m_v + (m - n)\phi_v, \quad \gamma_{10} = 0, \quad \delta_{10} = \phi_{uv},$$

$$\alpha_{01} = \Phi_v, \quad \beta_{01} = 0, \quad \gamma_{01} = m_v, \quad \delta_{01} = \phi_{vv},$$

$$\alpha_{11} = \Phi_{uv}, \quad \beta_{11} = \Phi_v + m\phi_{vv} + a m_v, \quad \gamma_{11} = b m_v + m_{uv}, \quad \delta_{11} = \phi_{uvv},$$

$$\alpha_{20} = \Phi_{uu} + p\{m_v + (m - n)\phi_v\},$$

$$\beta_{20} = 2\Phi_u + m_u \phi_v + 2m\phi_{uv} + a\{m_v + (m - n)\phi_v\},$$

$$\begin{aligned}\gamma_{20} &= \beta \{ m_N + (m-n) \phi_N \} , \\ \delta_{20} &= \phi_{uuv} + L \{ m_N + (m-n) \phi_N \} , \\ \alpha_{02} &= \Phi_{vv} + q m_N , \\ \beta_{02} &= \gamma m_N , \\ \gamma_{02} &= \Phi_N + m_{vv} + n \phi_{vv} + \delta m_N , \\ \delta_{02} &= \phi_{vvv} + N m_N .\end{aligned}$$

Hence the determinant

$$D_{ij} = (z_{ij}, z, z_{10}, z_{01})$$

has the value

$$\begin{aligned}D_{ij} &= D \{ \{ m_N + (m-n) \phi_N \} \{ m_N \phi_N \alpha_{ij} + (m_N \phi_{vv} + m_N \phi_N^2 - m_{vv} \phi_N) \gamma_{ij} - m_N \Phi \delta_{ij} \} \\ &\quad + m_N (m_N \phi_{uv} + m_u \phi_N^2 - m_{uv} \phi_N) \beta_{ij} \} ,\end{aligned}$$

where

$$D = (x, x_u, x_v, y) .$$

Therefore the asymptotic net on S_1 is given by

$$(88) \quad D_{20} du^2 + 2 D_{11} du dv + D_{02} dv^2 = 0 .$$

If

$$m_N + (m-n) \phi_N = 0 ,$$

the surface S_1 is identical with S_η ; under these circumstances, the net (88) reduces to

$$m_u du^2 + (m-n) \gamma dv^2 = 0 ,$$

which, as we have noted in §4, defines the asymptotic net on S_η . Again, if $\phi_N = 0$, the surface S_1 coincides with S_X , and the net (88) becomes

$$L du^2 + N dv^2 = 0 ,$$

which defines the asymptotics on S_X .

The developables of Γ_{xy} intersect S_1 in the curves of the net given by

$$(89) \quad du \, dv = 0.$$

The double rays of the involution determined at any point of S_1 by the asymptotic tangents and the parametric tangents through the point are defined analytically by equating to zero the jacobian of the quadratic forms (88) and (89):

$$(90) \quad D_{20} du^2 - D_{02} dv^2 = 0.$$

Hence the ruled surfaces of the congruence which intersect S_1 in the curves of the net (90) form a conjugate net of ruled surfaces, and this conjugate net has been defined geometrically in terms of the congruence.

VITA

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The writer desires to acknowledge his debt of gratitude to all the members of the staff of the Department of Mathematics and Astronomy at the University of Chicago, including former Professor F. R. Moulton and visiting Professors E. Bompiani, Dunham Jackson, A. B. Coble, and C. C. MacDuffee. The writer's grateful thanks are particularly extended to Professor E. P. Lane, who directed the writing of this thesis; his advice and encouragement were at all times most generously given. Acknowledgement is also due to Professors H. R.

Kingston, W. J. Patterson, and A. Woods, of the University of Western Ontario; their instruction first interested the writer in the science of mathematics.

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University of Chicago,

September, 1930.

